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THE DEVELOPMENT

OF

GREAT CIRCLE SAILING,

BY

G. W. LITTLEHALES,
U. S. HYDROGRAPHIC OFFICE.

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PREFACE TO THE FIRST EDITION.

This publication has for its object the furtherance of the effort of the Bureau of Navigation of the Navy Department to keep pace with the progress of the nautical sciences. It consists of an exposition of graphical and analytical methods embodying cardinal principles relating to the great circle, as applied to navigation, and gives publicity for the first time to several of the most convenient and useful methods yet devised.

The actual state of the science of great circle sailing is here presented, so as to give a clear conception of each method, and to furnish references where more extended information can be found.

GEORGE L. DYER,
Hydrographer.

PREFACE TO THE SECOND EDITION.

During the decade that has elapsed since the first edition of this publication was printed, a fuller recognition of the place of the great circle route in the problem of accelerating ocean transit has stimulated an advance to methods by which great circle courses can be taken from the Solar Azimuth Tables or measured from the chart compass with very great facility.

All the parts of the original work have been retained in this edition, and explanations of the new developments have been incorporated into those sections of the book in which they are appropriate.

J. E. CRAIG,
Captain, U. S. N., Hydrographer.

HYDROGRAPHIC OFFICE,
BUREAU OF EQUIPMENT,
May 1, 1899.

THE DEVELOPMENT OF GREAT CIRCLE SAILING.

INTRODUCTION.

It is not generally recognized that science, employing the mathematician and the engineer alike in the problem of shortening the duration of ocean transit, has accomplished as much by causing ships to travel fewer miles as by causing them to travel faster. In the age of steam propulsion the route of minimum distance in ocean transit is coming to take the place which was held by the route through the regions of favorable winds in the age of sail propulsion. A knowledge of the principles of the great circle must have been coeval with the knowledge of the spherical form of the earth, but in the early days of ocean navigation great circle sailing was doubly impracticable, for seamen were without the means of finding longitude and, moreover, the course of ships was controlled by the wind. Inasmuch as great circle courses alter continuously in proceeding along the track, it becomes necessary to know the latitude and longitude of the ship in order to determine the course to be followed. At the present day there are convenient means for determining at sea the longitude as well as the latitude with considerable precision, but in the early part of the present century these means did not exist and the principles of great circle sailing could not be applied.

Besides increasing the rate of travel, modern motive power, by making possible a departure from the old meteorological routes, has had another and a greater effect in the progress of the universal policy of civilized nations to accelerate transit from place to place to the utmost possible extent, because, under steam, even if they go no faster, ships may yet get farther toward the port of destination in a given time since they may be navigated along arcs of great circles of the earth. The increasing recognition among mariners of the sound principle of conducting a ship along the arc of the great circle joining the points of departure and destination, and the expanding sense of the advantages to be gained by a knowledge of this branch of nautical science, have greatly heightened the value of methods which place the benefits of the knowledge and use of the great circle track at the service of the mariner without the labor of the calculations which are necessary to find the series of courses to be steered. The general lack of

the application of the principles of the great circle in later times, and even in the present generation, seems to have resulted not from the want of recognizing that the shortest distance between any two places on the earth's surface is the distance along the arc of the great circle passing through them, nor that the great circle course is the only true course and that the courses in Mercator and parallel sailing are circuitous, but to the tedious operations which have been necessary and to the want of concise methods for rendering these benefits readily available.

A knowledge of the great circle route is important in working to windward with sailing ships, especially when far from the port of destination. A seaman not bearing in mind his great circle course, which is the only one that heads the vessel for her port, may unwittingly sail away from that port by taking the wrong tack. The great circle course to a far distant port may vary three or four points from the rhumb course. For example, on the route from Yokohama (Cape King) to Cape Flattery the great circle course at Cape King is NE., while the rhumb course is E. by N., a difference of three points. In this case for a wind directly ahead on the rhumb route an uninformed mariner would lay his vessel on either tack indifferently. If on the port tack, his vessel would head SE. by S., nine points away from the bearing of Cape Flattery. On the starboard tack she would head N. by E., only three points away. This is perhaps an extreme case, but it may serve to show how important it is that the master of a vessel should know his great circle course even when not distinctly pursuing a continuous great circle route.

It has been pointed out that the rhumb line, although appearing as a straight line on the Mercator chart, and thus giving a false idea that it represents the shortest route, is in reality a roundabout track, and that it is only when a vessel's course is shaped by the great circle passing through the places of departure and destination that she has the shortest possible distance to make and heads for her port as if it were in sight throughout the voyage.

In the case of the great circle track between Yokohama and Cape Flattery, the vertex, or point of highest latitude reached, is shown to lie within Bering Sea, and the track is consequently obstructed by the Aleutian Islands. It frequently occurs, when laying out an extended great circle route, that the lay of the land, or the extreme of climate, or dangers to navigation, when weighed in connection with the saving of distance that is made good on the great circle, leads the mariner to limit his track by a given parallel of latitude higher than which he decides not to go. Under these conditions the shortest route to follow is made up of an arc of the limiting parallel of latitude and two arcs of great circles which pass, respectively, through the points of departure and destination and whose vertices, or points of highest latitude reached, lie on the limiting parallel. In such a case the vessel's course is laid from the point of departure along the first great circle arc until its highest latitude, which is the latitude of the limiting parallel, is

reached; thence along the limiting parallel to the point of the highest latitude of the second great circle; and finally along the second great circle to the point of destination. Such a route is called composite.

With the exception of the miscellaneous methods of Airy, Chauvenet, Harris, Fisher, Sigsbee, and Proctor, the development of graphical methods in this branch of nautical science has proceeded in two distinct lines; the history of one is the history of the development of the principles of the gnomonic projection, and that of the other is an account of the various devices which have been contrived to find the vertex of a particular great circle.

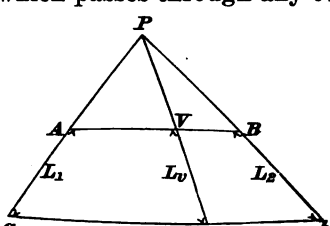
SECTION I.

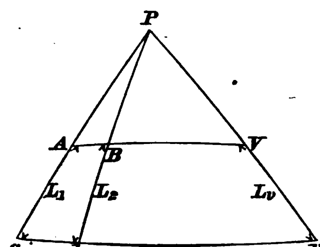
METHODS REQUIRING A KNOWLEDGE OF THE VERTEX.

By a system of great circles is meant all great circles whose common diameter is a diameter of the equator. If we consider such a system of circles to be drawn upon the surface of a sphere, and imagine the common diameter to revolve in the plane of the equator, all possible great circles will be described, for we have at first drawn all the great circles of one system and then revolved this system into all possible positions. From these considerations it is obvious that a distinguishing feature of every great circle is its inclination to the plane of the equator, or the latitude of its vertex. All properties of spherical courses, latitudes, and longitudes and distances from the vertex of a certain great circle of one system are identical in the corresponding great circle of every other system. Therefore, by tabulating these properties for each great circle of one system, tables of the properties of all great circles are formed.

TOWSON'S METHOD.

In 1850 Mr. John T. Towson, examiner to the Mercantile Marine Board at Liverpool, proposed a method for facilitating great circle sailing, consisting in the use of tables, such as described above, and a diagram, by means of which the whole operation of finding the successive spherical courses and distances on any great circle is reduced almost to an affair of inspection. These tables, as published by the British Admiralty, give the latitudes, spherical courses, and distances on great circles of the earth corresponding to each degree of longitude reckoned from the meridian of the great circle's vertex. Mr. Towson's diagram, the whole object of which is to find the vertex of the particular great circle which passes through any two places, is construeted as follows:

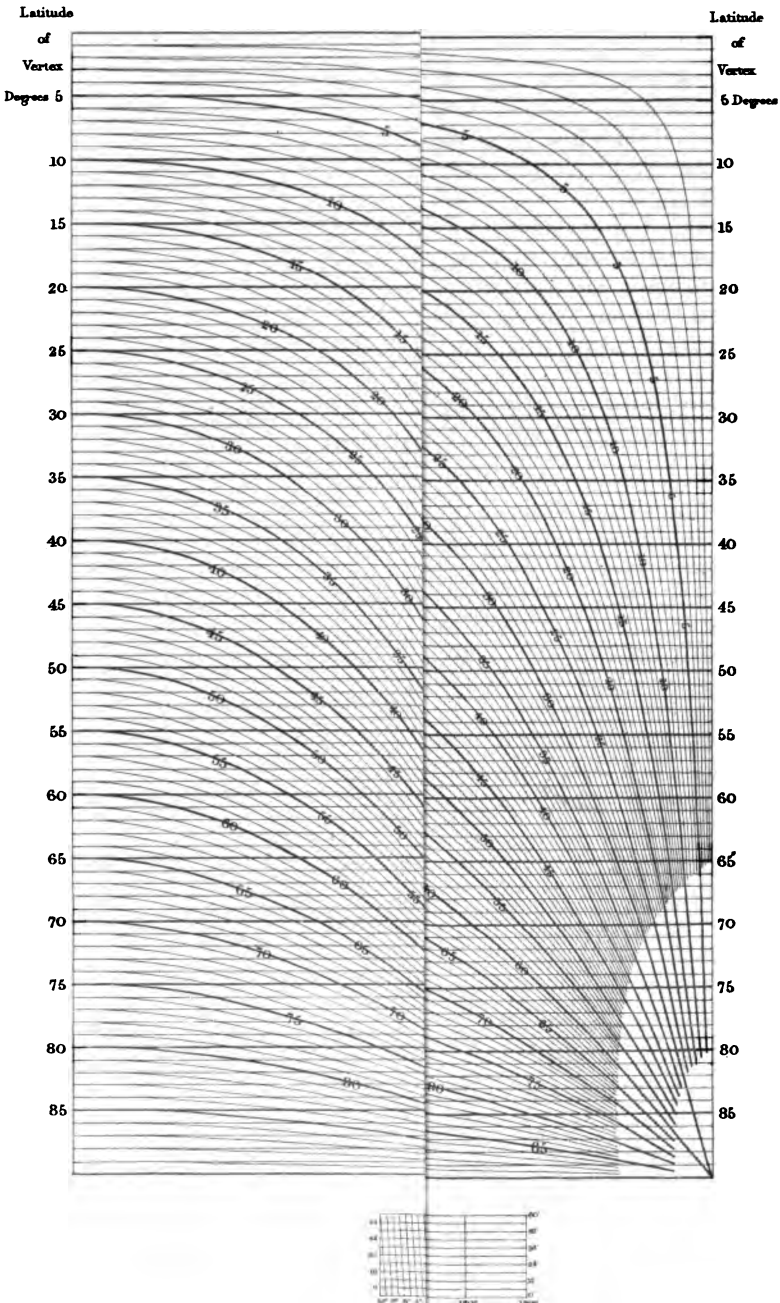




Let A and B represent two points on the surface of the globe, one being the place of departure and the other that of destination. Let P represent the pole, PA and PB the meridians passing through A and B. Let a great circle pass through A and B. The points a, v, b are on the equator. From P draw PV perpendicular to the great circle, then V will be the vertex.

In the spherical triangle APV,

$$\cos \mathrm{APV} = \frac{\tan (90 - \mathrm{L_v})}{\tan (90 - \mathrm{L_1})} \text{ or } \frac{\cot \mathrm{L_v}}{\cot \mathrm{L_1}} \qquad (1)$$


Latitude
of
Vertex
Degrees 5
10
15
20
25
30
35
40
45
50
55
60
65
70
75
80
85
Latitude
of
Vertex
5 Degrees
170°
180°

In the spherical triangle BPV,

$$\cos BPV = \frac{\cot L_v}{\cot L_2} \tag{2}$$

These two equations remain the same, whether the perpendicular falls within or without the triangle, for all possible values of the parts named. In either equation, any two of the terms being given, the third becomes known; and from the similarity of (1) and (2) it is obvious that each of these equations computed for all possible values of L_v and L_1 or L_2 will give the same series of results for the arc APV or BPV, and embraces all the values that the arc can have.

The successive values of this arc were computed for all the integral values of L_v and L_1 from 1° to 89°, inclusive. The results form the distance column of Towson's tables. They were also projected as ordinates to the axes named Meridians of Vertex in the linear index, which is appended, and the curves were drawn through the extremities of these ordinates.

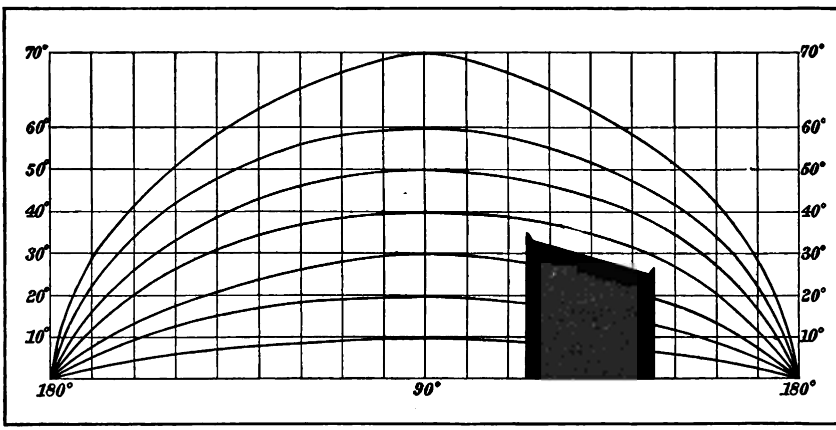
Suppose the vertex of the great circle passing through two places, A and B, is to be found; with a pair of dividers take out their difference of longitude from the scale of differences of longitude, and, if the latitudes of A and B have the same name, set one point of the dividers in the division of the diagram nearest to the right hand on that curve whose number corresponds to the latitude of either A or B, whichever is the nearer to the equator, and keeping the line joining the points of the dividers horizontal, follow the curve up or down till the other point meets the curve corresponding to the latitude of the other place. The index line passing through the points of the dividers in this position will indicate the latitude of the vertex. If that point of the dividers which stands on the curve corresponding to the latitude of departure be kept fixed, and the other be moved in a horizontal direction until the nearest meridian of vertex is reached, the dividers, being applied to the scale, will indicate the longitude of the point of departure from the vertex. With the latitude of the vertex and longitude from the vertex, thus found, the tables are entered and the distance from the vertex and the course are picked out. If the latitudes of A and B are of different names, the vertical line named Equator in the linear index must be kept between the points of the dividers in finding the latitude of the vertex, but the rest of the operation remains the same as before described.

DEICHMAN'S METHOD.

In 1857 Mr. A. H. Deichman, in full knowledge of and with a view to improving what had already been done by Towson, devised a diagram called a Scale of Great Circles to be used in connection with a table for finding great-circle courses and distances. The table is the same as Towson's, except that the longitudes and distances are reckoned from the intersection of the great circle with the equator, instead of the vertex. The Scale of Great Circles is a device for finding the vertex of any great circle, which is all that is necessary, for when the vertex of a great circle is known all courses and distances on that great circle are virtually known. Deichman's diagram or Scale of Great Circles con-

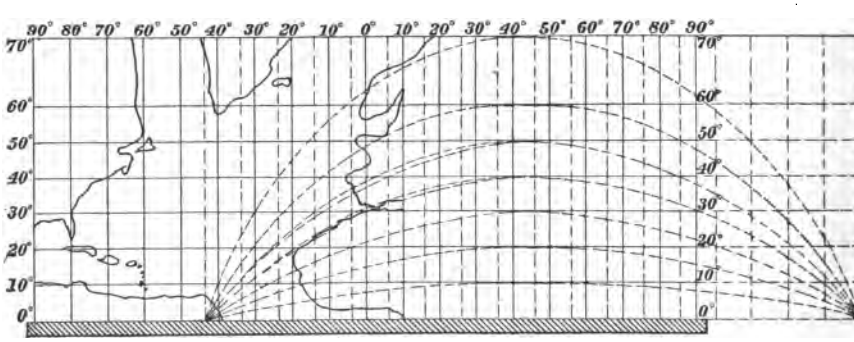
sists of a Mercator projection, of such scale as to make the representation of one-fourth of the earth's surface of convenient size, with a series of great circles projected upon it.

His method of finding the vertex consists in cutting out of paper or card-board, on the same scale as that of the Scale of Great Circles, an "Index Model," whose two upper edges shall represent the positions of the two places through which the great circle passes in their approximate latitudes and longitudes, and sliding this (see figure) over the Scale of Great Circles until both the upper edges shall lie on the same great circle. The latitude of the vertex is then read off from the Scale of Great Circles.



BREVOORT'S METHOD.

Mr. J. Carson Brevoort submitted to the Hydrographic Office, for an opinion, in July, 1887, a method for facilitating great circle sailing which closely resembles the methods of Towson and Deichman. His tables are the same as Towson's, except that the column of distances is omitted and a column of compass courses, reading to one-eighth of a



point, is added. The index diagram is a Mercator projection, laid down on transparent material, with a system of great circles projected

upon it, like Deichman's Scale of Great Circles. This is designed to slide over a Mercator chart on the same scale as that of the index diagram, with the equators of the diagram and the chart in coincidence, until the two places between which the great circle track passes lie on the same great circle of the diagram. In the figure the lines represented by dashes are those which occur on the transparent diagram. It will thus be seen that, while Deichman's method causes any part of the earth to revolve over one system of great circles, Brevoort's causes one system of great circles to revolve over any part of the earth.

BERGEN'S METHOD.

In 1857 W. C. Bergen, master in the British mercantile marine, published a set of spherical tables of general adaptability to the problems of nautical astronomy, and a diagram, by the aid of which the results in the tables are rendered applicable to the problem of great circle sailing.

Although this method can not be used with the same degree of facility as some of the others of the same character, it is well worth attention on account of the radical principles involved.

The diagram, which is appended, represents a quadrant of the earth's surface with the parallels of latitude and the meridians of longitude from degree to degree. Each fifth parallel and fifth meridian is numbered and drawn stronger than the others, and in order further to distinguish them the alternate parallels between them are dotted. Over this net-work of parallels and meridians a system of great circles is drawn.

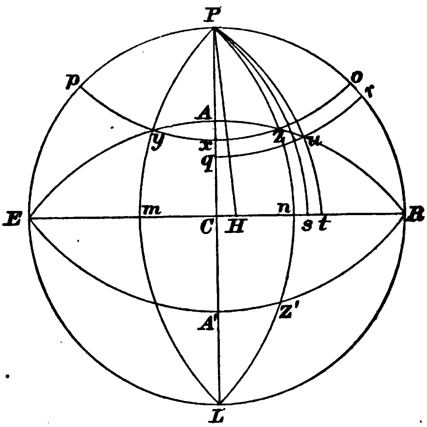
Under the equator the degrees from the meridian of the vertex are numbered, thus showing the longitude from the vertex, and the half difference of longitude; and under this again these numbers are doubled, indicating the difference of longitude; and, to obviate the necessity of taking the supplement of the difference of longitude, in the case of the places being on opposite sides of the equator, the third line of numbers is given, containing the difference of longitude reckoned from the point where the great circle crosses the equator.

The object of the diagram is to find the latitude of the vertex of the particular great circle which passes through the points of departure and destination, and the longitude of the point of departure from the meridian passing through the vertex.

In the figure, let C, the center of the primitive PELR, be the projection of a point on the earth's equator, PL that of a meridian passing through the point C, and ER that of the equator. Let P represent the nearer pole, and P*m*L and P*n*L meridian circles equally inclined to the primitive. Then, it is evident that PL bisects the angle *m*P*n*, and therefore *m*P*n* is double the angle CP*n*.

Again, let a great circle EAR be drawn through ER, cutting P*m*, PL, and P*n* in points *y*, A, *z*. And through *yz* let the parallel of latitude *pyxzo* be drawn.

Now, if the quadrant PCE be conceived to turn around the radius PC, which remains fixed, and to be placed on the quadrant PCR, the projections of the one will fall upon the corresponding projections of the other, and will coincide with them; P*m* will therefore coincide with P*n*, AE with AR, *px* with *ox*, and the point *y* with the point *z*.

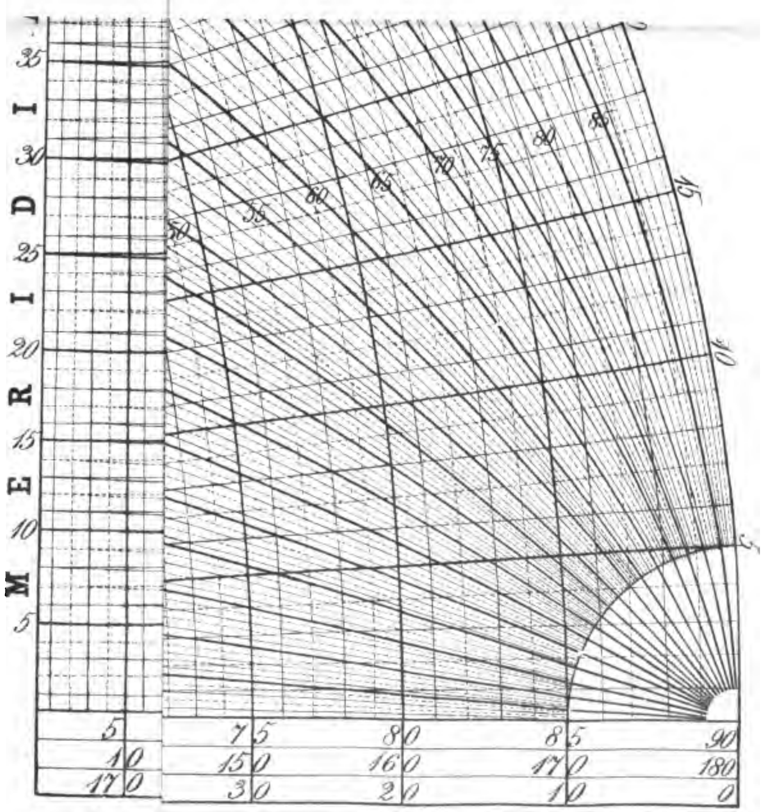


A is the vertex of the great circle EAR, PL is the meridian of the vertex, and CP*n* is the longitude of the point *z* from the vertex.

Hence, when two places are given on the same parallel of latitude, to find the latitude of the vertex and the longitude from the vertex, let C*n* be equal to half the difference of longitude, and *ox* represent part of the parallel of latitude passing through the places; with a pointed instrument trace up the meridian P*n*, and with another trace along the parallel *ox* until the curves meet in the point *z*, then trace the great circle R*z*AE, passing through *z*, until it meets the meridian of vertex in the point A. Then CA is the latitude of the vertex, and C*n* is the longitude from the vertex.

Again, if the places are on the same side of the equator and on different parallels of latitude, *ox* and *rq*, let P*n*, P*s*, and P*t* represent meridians equidistant from one another, and let C*s* be equal to half the difference of longitude. With the right hand trace up the meridian P*s* until it cuts the lower parallel of latitude, *rq*, then, with the left hand, trace it up again until it meets the higher parallel of latitude *ox*, then trace the lower latitude to the right hand and the higher to the left, being careful to move an equal number of degrees on each side of the meridian P*s*, until the curve of the great circle passes through both points. Let these points be represented by *z* and *u*, and let R*uz*A represent the great circle, then AC will represent the latitude of the vertex, and C*n* and C*t* the longitudes of the points *z* and *u* from the vertex.

It is evident that if *y* be the place of the ship and *u* that of her destination, or conversely, the meridian of vertex falls between them, and


MERIDI

that *mn* added to *nt* is the difference of longitude; but C*n* is equal to half of *mn*, and therefore twice C*n* added to *nt* is equal to the difference of longitude. Now, *ns* is half of *nt*, and when twice C*s* is considered as the difference of longitude, when the hand is moved to the left from *s* to *n*, twice *ns* is taken from the difference of longitude, but the other hand being moved from *s* to *t*, they are distant from each other *nt*, which is equal to twice *ns*, and *nt* being added to twice C*n*, the difference of longitude is the same as at first.

If the meridian of the vertex fall outside of the portion of the great circle track which lies between the points of departure and destination, as in the case in which *z* and *u* are taken to represent those points, let CH be equal to *ns*, which is equal to the half difference of longitude between *z* and *u*, and let PH be traced up as before. In this case the left hand will meet the meridian of vertex without meeting the great circle, and the right hand must be moved until it is at a distance from the meridian of the vertex equal to the difference of longitude. Both hands must then be moved equally until they arrive at the points *z* and *u* where the great circle passes through them.

For the explanation of the case in which the points of departure and destination are on opposite sides of the equator, imagine the opposite pole of the primitive to be taken as the projecting point, and equal circles to be projected upon the lower right hand quadrant LCR, and conceive it to be turned upon the radius CR as an axis until it shall coincide with the quadrant PCR. The projections of one will then coincide with the corresponding projections of the other; and if the places are in equal latitudes, as *z* and *z*′, then *n*R will be half the difference of longitude, and C*n* will be half its supplement. Therefore by taking the supplement of the difference of longitude and proceeding exactly as in the other cases the latitude of the vertex AC or A′C and the longitude of the vertex C*n* or C*t* will be found.

The latitude of the vertex and the longitude from the vertex having been found from the diagram by inspection, a table, of which the following is a specimen page heading, is entered, and the course and the distance from the vertex are picked out.

Latitude of vertex, 19°.

Longitude from vertex.	Distance from vertex.	Course.				
1°	0° 57′	0° 19′				

In the actual construction of the tables the author greatly abridged his work by observing the trigonometric principle that, if the complements of the hypotenuse and base of a right-angled spherical tri-

angle be taken as the base and hypotenuse of another to the same angle, the perpendicular of each triangle is equal to the complement of the arc which measures the remaining angle of the other triangle.

Let the figure represent a projection on the plane of the primitive. Let P and P′ be the poles, EQ the equator, EVQV′ any great circle, whose northern and southern vertices are at V and V′, respectively, and PLMP′ the meridian passing through the point L on the great circle EVQV′, at which the course and distance are required.

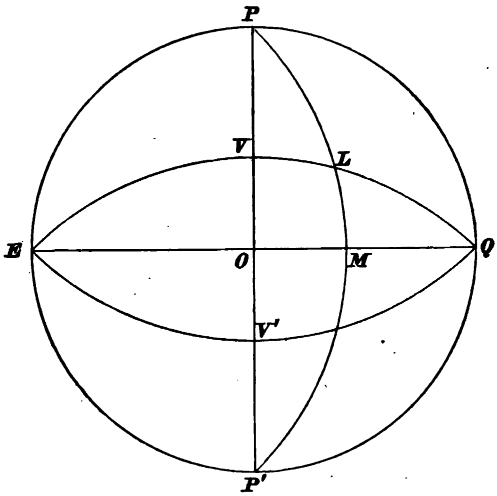
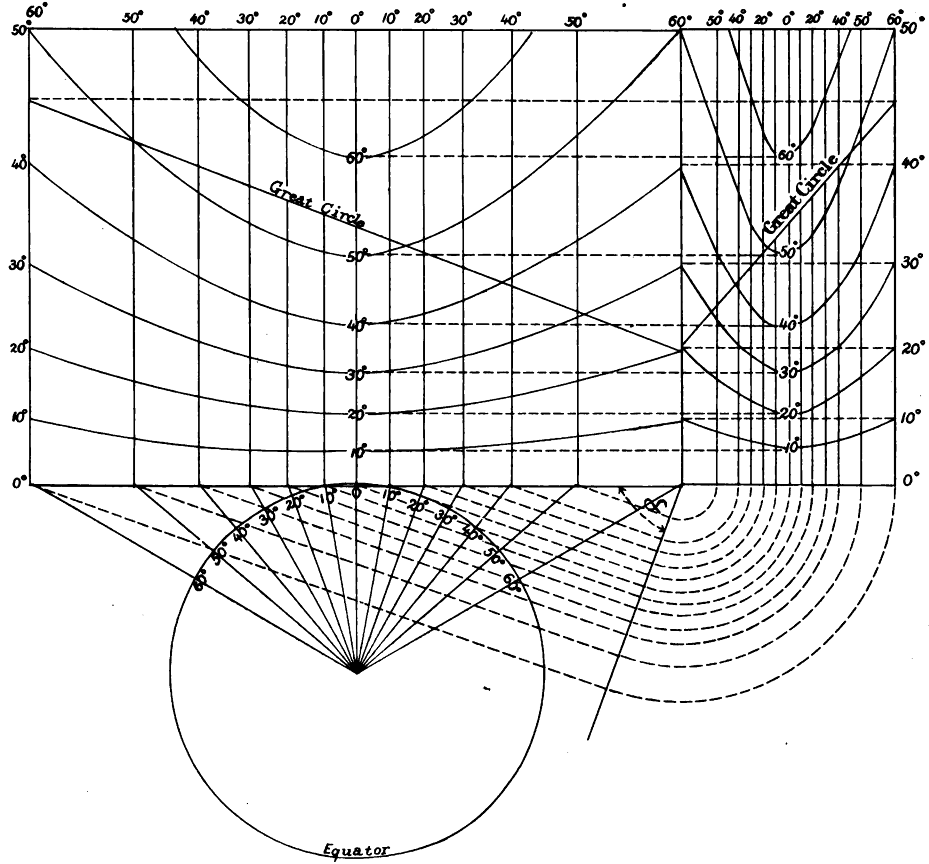


Fig. 19s.

The angle LQM, between the plane of the equator and the plane of the great circle, is numerically equal to the arc VO, or the latitude of the vertex V. In the triangle LMQ, the side LQ is the complement of the distance of the point L from the vertex, the side MQ is the complement of the longitude of L from the vertex, the side LM is a measure of the latitude of L, the angle MLQ represents the course, and the angle LQM the latitude of the vertex. In the construction of the tables, the triangle LMQ was computed to every degree of the angle Q, and also of the arc LQ, until LV was equal to or next less than MQ. Then the complements of LQ and MQ were taken, and the trigonometric principle already mentioned was applied.

With regard to the degree of dependence to be placed upon this method, Mr. Bergen remarks, in his writings on the subject, that, by means of an extensive induction, he arrived at the conclusion that for a difference of longitude equal to about 15°, the course may be depended upon generally to less than one-eighth point; in some instances, when the difference of longitude is small and the latitude high, the course may be erroneous nearly one-quarter of a point; but this will not happen with a difference of longitude more than 15° and a latitude less than 60°.

Gnomonic Projection. Contracted.



THE TERRESTRIAL GLOBE.

The strict practical solution of the problem of great circle sailing, which consists in using a terrestrial globe, is of equal convenience and of at least equal accuracy with the foregoing method. A great circle passing through any two places upon the earth's surface may be traced upon a terrestrial globe by elevating or depressing the polar axis and, at the same time, turning the globe until the places coincide with the upper edge of the wooden horizon. In this position of the globe the upper edge of the wooden horizon represents the great circle, and can be used as a ruler for tracing it upon the surface of the globe.

Having thus traced upon the globe the great circle passing through the two given places, their distance apart is shown by the number of degrees intercepted between them on the wooden horizon, the latitude and longitude of the vertex can be read off, and the various places through which the great circle passes are shown.

It will thus be seen that, instead of finding the latitude of the vertex and longitude from the vertex by the diagrams which have been devised by Towson, Deichman, Bergen, and Brevoort, a terrestrial globe can be used for finding these elements with quite as much efficiency. After which the spherical tables, which have been arranged by any of these authors can be used for finding the courses and distances.

Or the latitudes and longitudes of a sufficient number of points or places on the track of a great circle, thus traced upon a terrestrial globe, may be read off and the corresponding positions plotted on a Mercator sailing chart. Then, connecting the points by drawing a curved line through them, a graphical representation of a great circle upon the chart is obtained without any computation.

THE DIRECT TRACK SCALE.

This was recently designed by Mr. Gustave Herrle, of the U. S. Hydrographic Office, as an appendage to the Mercator chart, with a view of making the chart more directly available in pursuing a great-circle track. It consists of a gnomonic projection, with the point of contact on the equator, contracted in longitude. From this the position of any point of a great-circle track can be read off and transferred to the Mercator chart. There are also logarithmic scales of sines and cosines for the measurement of courses and distances.

The gnomonic projection, with the point of contact on the equator, is that which is used by Lieutenant Hilleret, French navy, whose great circle sailing charts will be referred to.

Let r represent the radius of the sphere;

θ represent the longitude of any meridian reckoned from the meridian passing through the point of contact, or, as it is generally called, the middle meridian;

φ represent the latitude of any parallel.

Then the successive distances of the parallel straight lines of the pro-

jection, which represent the meridians, from the middle meridian, will be $r \,.\, \tan \theta$; and the points at which the successive parallels of latitude cut the middle meridian will be $r \,.\, \tan \varphi$, measured from the equator.

The distance above the equator at which any parallel whose latitude is φ cuts a meridian whose longitude from the middle meridian is θ will be $r \,.\, \tan \varphi \,.\, \sec \theta$.

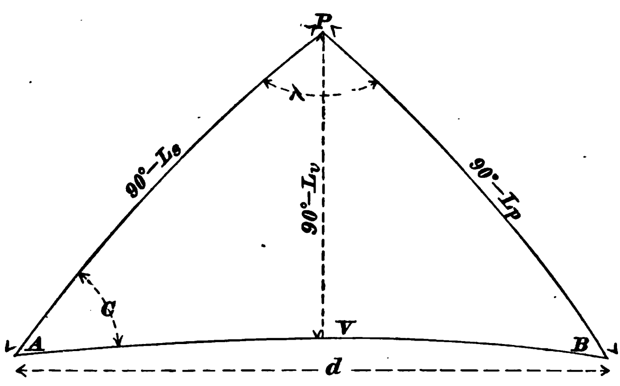
By the application of the preceding expressions the gnomonic equatorial projection of the Northern Hemisphere up to latitude 60°, shown in the above figure, has been constructed.

In order to economize area and yet preserve all the properties of the projection in relation to direct tracks the first projection is projected orthographically upon a second plane, which passes through a meridian of the first plane of projection and makes with it any angle, α.

The degree of contraction will then be directly proportional to the cosine of the angle α, so that the contracted projection could be constructed at once by inserting in the expression for distances of the successive meridians from the middle meridian the cosine of the angle of inclination of the two projecting planes. This expression would therefore be $r \,.\, \tan \theta \,.\, \cos \alpha$.

The expression for constructing the parallels of latitude will remain the same, *i. e.*, $r \,.\, \tan \varphi \sec \theta$.

THE MEASUREMENT OF COURSES AND DISTANCES ON THE DIRECT TRACK SCALE.



In the figure, let AB represent any great circle of the earth, A the place of departure or the ship's position at any time, and B the place of destination.

Let C represent the course at A,

V represent the vertex of the great circle AVB,

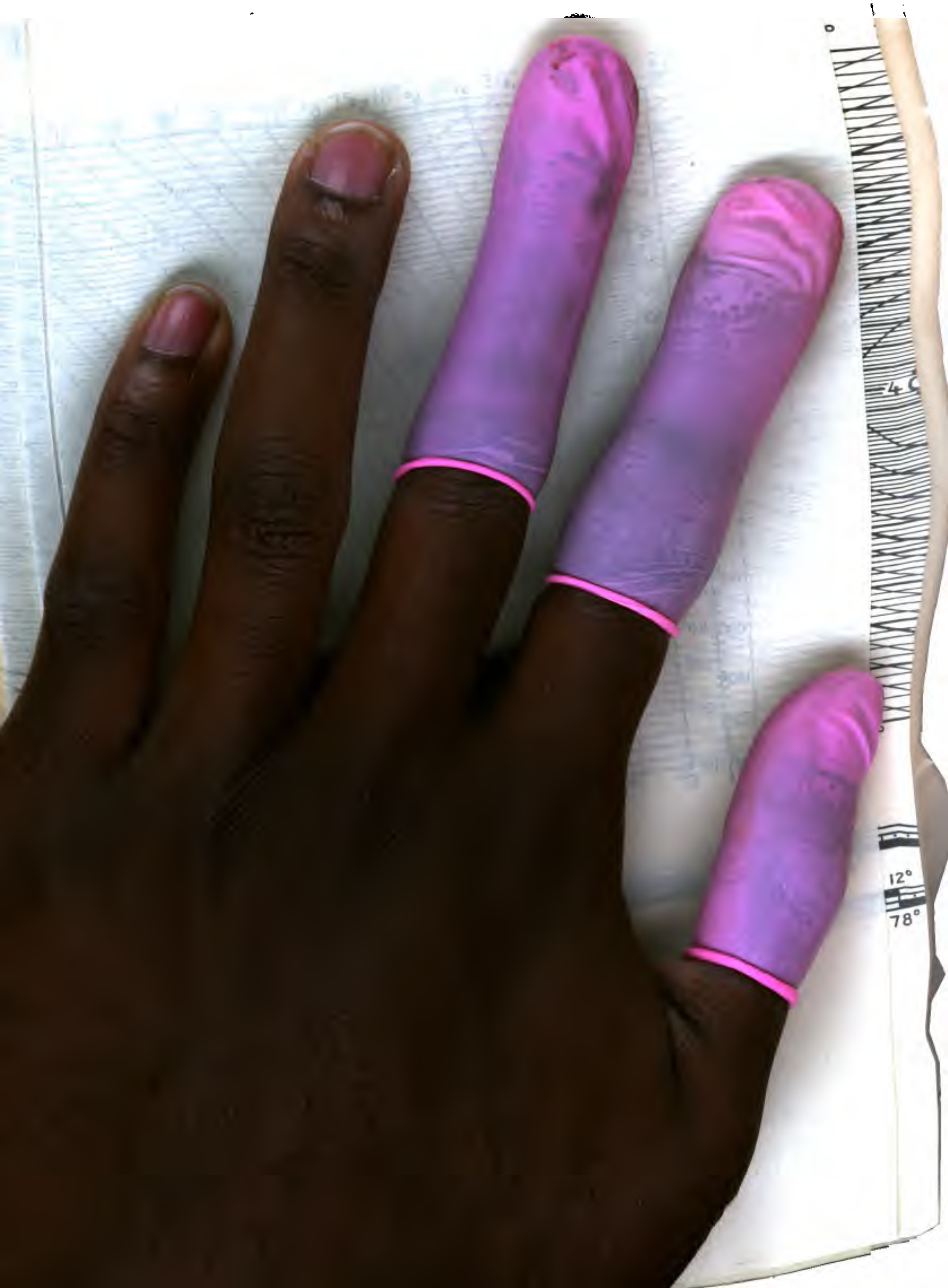
d represent the distance between A and B,

L_s represent the latitude of A,

L_v represent the latitude of V,

L_p represent the latitude of B,

λ represent the difference of longitude between A and B.


12°
78°

From the right-angled spherical triangle APV,

$$\sin C : \sin 90^\circ :: \cos L_v : \cos L_s$$

$$\sin C = \frac{\cos L_v}{\cos L_s}$$

$$\log \sin C = \log \cos L_v - \log \cos L_s \qquad (1)$$

From the spherical triangle APB,

$$\sin C : \sin \lambda :: \cos L_p : \sin d$$

$$\log \sin d = \log \sin \lambda + \log \cos L_p - \log \sin C. \qquad (2)$$

In order to solve the logarithmic equations (1) and (2) graphically, and thus find the course at any point and the distance from that point to the place of destination without computation, a length is adopted corresponding to a certain number of units of the mantissa of a log sin or log cos, and a linear scale is thus constructed giving the necessary angles.

The Direct Track Scale and its accompanying logarithmic scales are here shown reduced to one-half the size which it is proposed to use in practice.

The following rules for using this method are given by Mr. Herrle:

On the meridian of the diagram marked 0°, plot the latitude of the place of departure, and on the meridian whose longitude from the meridian marked 0°, is equal to the difference of longitude between the places of departure and destination, plot the latitude of the place of destination, connect the two points thus plotted by a straight line and find by inspection on the diagram the highest latitude which it reaches. This latitude will be L_v, the latitude of the vertex of the great circle which passes through the places of departure and destination.

Find L_s and L_v by the lower readings on the logarithmic scale; take their difference and lay it off from the right end of the scale toward the left. The upper reading of the point laid off corresponds to C.

Courses greater than 28° may be measured on either scale, courses less than 28° on the lower scale only.

For the measurement of distances, find C at L_s and λ on the upper readings and L_p on the lower readings; take the difference between C and λ and lay it off from L_p toward the right or left, according as the place of destination is to the right or left of the ship's position. The upper reading of the point laid off corresponds to d, the distance.

Distances less than 28° or 1,680 nautical miles can be measured on the lower scale only.

SECTION II.

METHODS DEPENDING UPON THE GNOMONIC CHART.

The object of charts is to exhibit, by suitable representation, on a reduced scale and on a plane surface, the relative positions of points, lines, or objects on the earth's surface; and since such positions are usually defined by spherical coordinates, the primary object of the projection upon which the chart is based is the delineation of these circles of reference according to certain assumed or fixed geometrical laws. Any point, line, or object intended for representation may then be laid down by means of its known coördinates, and, conversely, the coordinates of any plotted point may be ascertained. The gnomonic chart is based upon a system of projection in which the plane of projection is tangent to the surface of the sphere, and the eye of the spectator is supposed to be situated at the center of the sphere, where, being at once in the plane of every great circle, it will see these circles projected as straight lines where the visual rays, passing through them, intersect the plane of projection.

GODFRAY'S GNOMONIC CHART.

In 1858 Hugh Godfray, M. A., Fellow of the Cambridge Phil. Soc., prepared, for the purpose of great circle sailing, two gnomonic charts covering the greater part of the world—one for the Northern and one for the Southern Hemisphere. He used as points of contact the geographical poles of the earth.

The parallels of latitude are therefore represented by a series of concentric circles whose radii are equal to r tan $(90° - \text{lat.})$ or r cot lat., where r is any convenient linear magnitude, and represents the radius of the sphere. The meridians are straight lines drawn from the common center or pole, dividing each circumference into three hundred and sixty equal parts. Any one of these being selected for the prime meridian, the coast lines of different countries may then be traced in the usual manner by means of the longitudes from the prime meridian

and the latitudes of the different points. Mr. Godfray also devised means tending toward convenience for the measurement of courses and distances on the great circle track. His diagram for this purpose, constructed upon a separate sheet, measuring 15 by 19 inches, consists of a series of concentric curves corresponding to the parallels of latitude, bounded by a vertical line, AC, and a horizontal line, AB. The horizontal line is divided into seventy equal parts, representing the degrees of latitude from 20° to 90°.

The vertical line AC is a scale of distances from the highest latitude and is divided into twenty-one equal parts, each representing 200 nautical miles. Through the various points of division of these scales are drawn horizontal and vertical straight lines over the whole diagram.

The concentric curves corresponding to the parallels of latitude are traced from the following considerations:

Let ABC represent any great circle,

B represent its vertex,

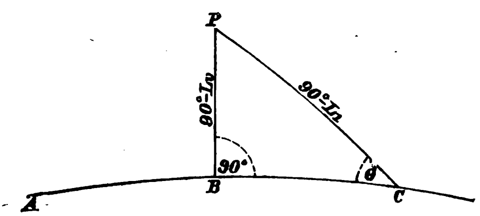
P represent the pole,

L_v represent the latitude of the vertex,

L_1 represent any lower latitude on the great circle,

d represent the distance from the vertex,

we have from the right-angled spherical triangle PBC, by Napier's analogies:



$$\cos (90^\circ - L_1) = \cos d \cos (90^\circ - L_v)$$

$$\cos d = \cos (90^\circ - L_1) \sec (90^\circ - L_v) = \sin L_1 \operatorname{cosec} L_v$$

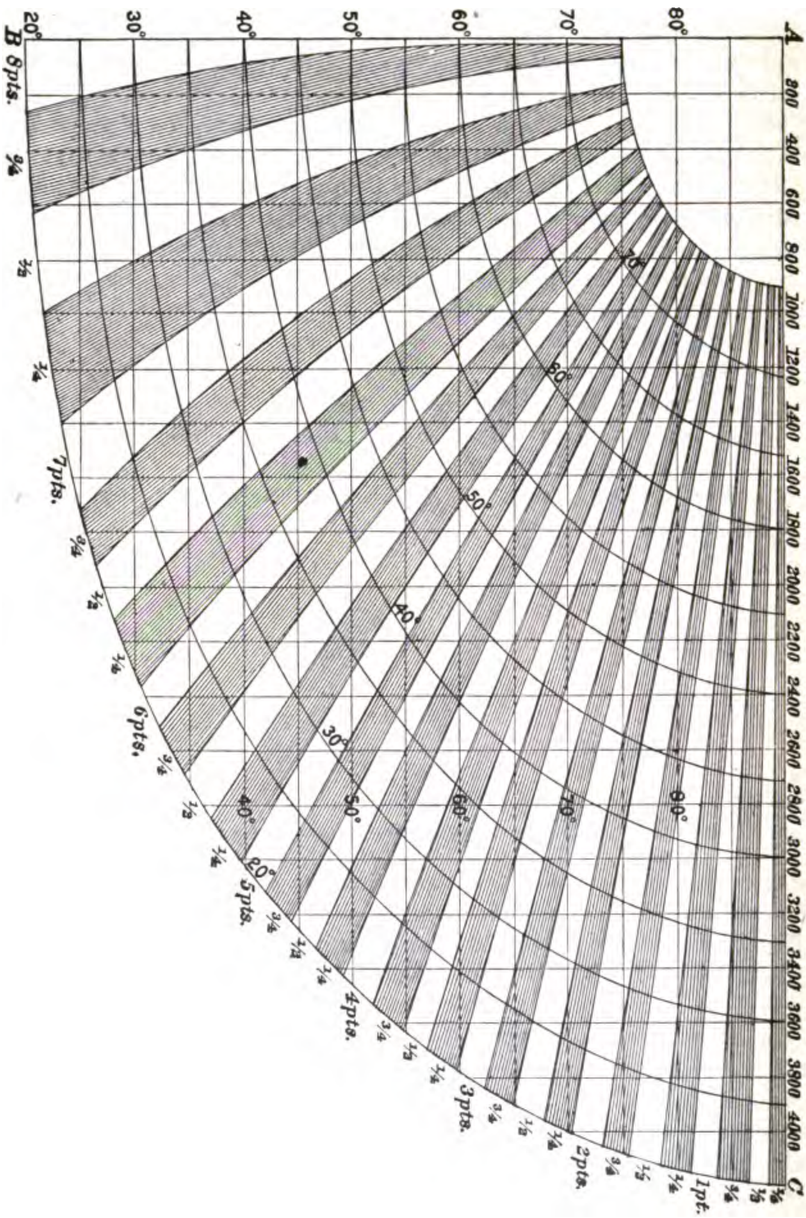
which determines the distance d at which the latitude curve L_1 crosses the horizontal line passing through the division L_v.

Thirty-two curved lines are drawn radially from the point marking 90° latitude and zero distance. The spaces between these are alternately light and shaded, and are marked in points and quarter-points for the determination of courses.

At the vertex of a great circle the course is due east or west, and as we move along the great circle away from the vertex the course will alter continuously as the distance alters. The connection between the course and distance will be determined by reference to the right-angled spherical triangle in which θ represents the course at any point whose distance is d from the vertex. From this triangle, $\sin d = \cot \theta \cot L_v$, which determines the distance of the point where the course is θ; and the curve for the course θ in the diagram must pass through that point

where the vertical line corresponding to the highest latitude L_v is met by the horizontal line at the distance *d*.

GODFRAY'S COURSE AND DISTANCE DIAGRAM.



The following example will illustrate the use of the diagram: Suppose it is required to find the successive courses and distances to

be run on each course between two places, A and B. Draw a straight line connecting A and B on the gnomonic chart, and observe the highest latitude reached by this line between A and B, or produced. Suppose the highest latitude is found to be 44½° S. and the latitude of A, the point of departure, is 35° S. Now, refer to the diagram and find the point where the vertical line through 44½° is crossed by the 35° curve. This falls on the light space corresponding to 6 points. Hence the first course is S. 67½° E. or W., according as B is east or west of A. Then, proceeding along the vertical line 44½° toward the highest latitude, measure the breadths of the successive light and shaded spaces, and the required series of courses and distances will be obtained.

KNORR'S GNOMONIC CHART.

In 1869 Mr. E. R. Knorr, of the U. S. Hydrographic Office, designed a chart of the North Atlantic ocean on the gnomonic projection, with a view to exhibiting at once all the influences controlling the navigation of that ocean. To this end his chart embraces the hydrography of the ocean, the topography of its shores and islands, the forces and directions of the winds which may be expected at different seasons, the general flow of the currents, and all great circle arcs, with means for finding courses upon them. The plane of projection is tangent to the sphere in latitude 30° north and longitude 40° west. The meridians of longitude are therefore represented by a series of straight lines converging toward the north, and the parallels of latitude by a series of elliptic and hyperbolic curves. The shortest distance between any two points on this chart is the distance on a straight line joining them, since, according to the principles of the gnomonic projection, every straight line represents a portion of the circumference of a great circle. For the measurement of courses a contracted form of Mr. John T. Towson's tables is prepared upon the face of the chart. These tables are fully treated of in the reference to Towson's method. The latitude of the vertex of the great circle track, which is required in finding the courses from these tables, is readily found by inspection on the chart in all ordinary cases. For those cases in which the vertex of the great circle falls beyond the upper border of the chart simple means of considerable convenience are given for finding the vertex. The author gives a rule for computing great circle distances, and states in a note on the chart that distances can not be directly measured on the gnomonic projection. This chart was never published, because in its chief feature, that of facilitating great circle sailing, it was quickly supplanted by better methods.

HILLERET'S GNOMONIC CHART.

In 1879 Lieutenant Hilleret, French navy, designed, for the purpose of great circle sailing, a set of gnomonic charts, with the point of con-

tact on the equator. The parallels of latitude are therefore represented by a series of hyperbolas whose transverse axes are equal to r. tan. latitude, and whose conjugate axes are equal to r, where r is any convenient linear magnitude and represents the radius of the sphere. The meridians are parallel straight lines whose distances from the meridian passing through the point of contact are equal to r. tan. longitude, reckoned from the point of contact. For the measurement of courses there is added a Mercator projection covering 10° of longitude, and of the same extent in latitude as the northern or southern portion of the chart. The great circle track being drawn upon the chart, the latitude of the point of departure and that of a point two or three degrees of longitude from the point of departure toward the point of destination are read from the chart. These points are then plotted on the Mercator projection in their respective latitudes and with their correct difference of longitude, and the bearing of the line joining them, as measured by means of a protractor, is read off for the first course. The succeeding courses are determined in the same manner. This method is approximate. The direct means for the measurement of distances exist in the projection. Since in this projection all great circles which are equally distant from the point of contact must each have an angular unit represented by the same linear magnitude, and since the meridians are great circles, to determine the length of any great circle track we have simply to let fall a perpendicular from the point of contact to the track, lay off this perpendicular distance from the point of contact along the equator, and draw a meridian through the extremity of it. The portion of the great circle track on one side of the foot of the perpendicular being laid off above the equator, and that on the other side below the equator, the difference of latitude between the two points thus marked will be the required distance.

JENZEN'S GNOMONIC CHART.

Mr. Carl Jenzen, master mariner, presented for examination at the Hydrographic Office, Navy Department, in July, 1887, a gnomonic chart of the North Atlantic Ocean, which combined, though without his knowledge, the method of projection used by Mr. Godfray in 1858 with the method of measuring courses which was adopted and published by the French in 1879. There exists a slight difference between the course diagrams of Lieutenant Hilleret and Mr. Jenzen. The former used a pure Mercator projection while the latter preserves an equality in the degrees of latitude throughout his diagram and causes the meridians to converge in the ratio of the cosine of the latitude. Both methods are approximate. Of all the available points of contact on the sphere, considered with reference to their adaptation to modern wants in great circle sailing, the geographical pole is least adapted. No complete gnomonic chart of the Indian Ocean could be made on this plan, no

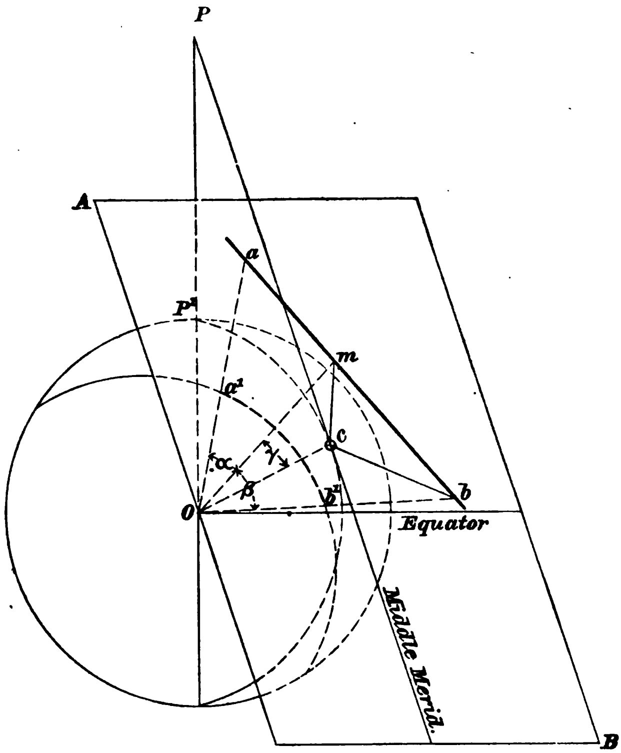
means could be devised by which the advantages of the great circle route could be applied in passing from one polar hemisphere to the other. Inasmuch as the radius with which each parallel is described in the gnomonic projection with the point of contact at the pole is equal to the tangent of the co-latitude of that parallel, and as the co latitude of the equator is 90°, the radius with which the equator would be described is tan $90° = \infty$. The delineation of the equator and of regions near the equator is therefore impossible when the point of contact is the pole.

HERRLE'S METHOD.

Within the present century many attempts have been made so to devise the gnomonic projection as to treat conveniently the problem of great circle sailing. Various successful applications were made from time to time, as seen in the charts of Mr. Godfray and Lieutenant Hilleret. No one succeeded, however, in making the projection itself an embodiment of the means for measuring both courses and distances until Mr. Gustave Herrle designed his great circle compass and suggested the application of the perpendicular distance from the point of contact to the great circle track as an argument in measuring distances on the chart itself.

The principles of the Mercator projection are such that great circles on the sphere will not generally appear as straight lines on the chart, but any straight line on the chart, excepting a meridian of longitude or a parallel of latitude, represents a rhumb-line and indicates a particular mutual bearing of two places so connected. On the sphere such a line is known as a loxodromic curve, and it possesses the property of cutting the meridians at equal angles, so that, in pursuing a straight line on the Mercator chart, the course is constant, though the route is really circuitous. The simplicity of the methods necessary for navigating this circuitous track and the long duration of its usage have so intrenched it in the estimation of mariners that no method of handling charts not analogous to these has found favor with them.

Another essential consideration in the construction of charts for great circle sailing is a method that affords facilities for measuring the course and distance from the actual position of the vessel independently of any great circle track that may have been previously laid down, just as the rhumb course and distance are measured on the Mercator chart from the actual position in which the vessel is found to be. Both of these principles are recognized by Mr. Herrle in the construction of his great circle sailing charts. With a notable degree of public spirit he gave to the Hydrographic Office his plans, as they existed in 1881, and he has contributed largely to the subsequent maturing of them and their publication in the form of the present excellent great circle sailing charts issued by the Hydrographic Office.



Let the plane AB represent the plane of projection tangent to the sphere whose center is O at the point c in lat. φ, and long. λ. Let $a'b'$ represent any great circle on the sphere, ab its projection on the tangent plane.

$Oc=r$ is a radius of the sphere and is perpendicular to the plane AB The angular distance of a from b is obviously equal to the angle at O, included between the lines Oa and Ob.

Draw Om perpendicular to ab. This will divide the triangle Oab into two right-angled triangles, Oam and Obm, in which

$am=Om \tan \alpha$, and $bm=Om \tan \beta$, or $am+bm=ab=Om (\tan \alpha+\tan \beta)$.

Draw cm, connecting the point of tangency c with the foot of the perpendicular Om, and let γ represent the angle subtended by cm at the center of the sphere. Since Oc is perpendicular to the plane of projection it will be perpendicular to cm, which is a line in that plane, therefore

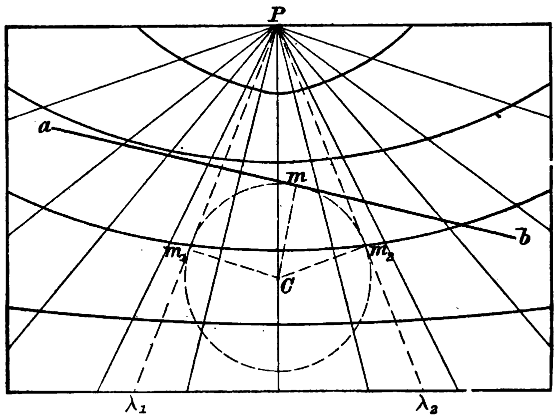
$$Om=r \sec \gamma \qquad (2)$$

Substituting the value of Om in (1) it becomes

$$am+bm=ab=r \sec \gamma (\tan \alpha+\tan \beta) \qquad (3)$$

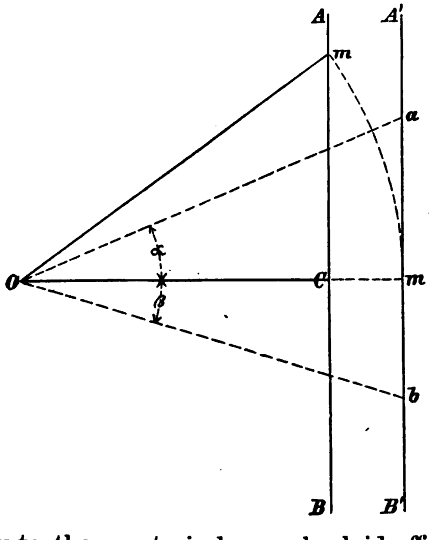
From equation (3) it appears that on the gnomonic projection the length of every great circle arc, and consequently every arc of a meridian, is equal to the algebraic sum of two tangents corresponding to a

certain radius, $Om = r \sec \gamma$, in which γ is the angle, at the center of the sphere, which is subtended by the perpendicular from the point of contact upon the projected great circle arc. Let a great circle, *ab*, be drawn in any position on the chart (figure), mark the point *m*, and with



c as a center and *cm* as a radius describe a circumference. The two meridians in longitude λ_1, and λ_2 will be tangent to this circumference at m_1, and m_2, respectively. These three great circles, *ab*, $P\lambda_1$, and $P\lambda_2$, have this principle in common, that a certain length laid off on each from m, m_1, and m_2 will correspond to the same angular distance on all three. This principle will also be true with reference to any other great circle track drawn tangent to the circumference m, m_1, m_2.

If a line of tangents, AB, be drawn corresponding to the radius $Oc = r$, the assumed radius of the sphere upon which the projection is constructed, and the distance *cm*, the perpendicular distance from the



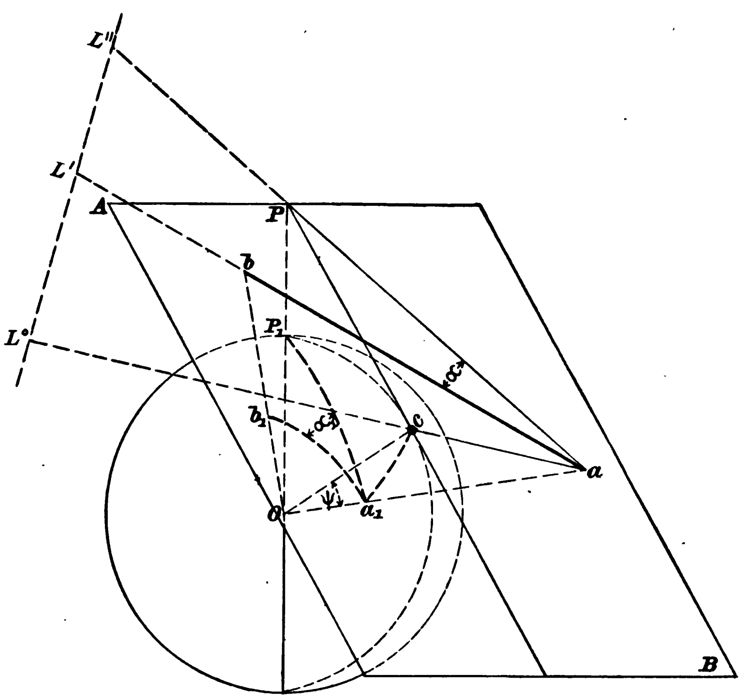
point of tangency to the great circle arc, be laid off upon it, we shall have the true length of $Om = r \sec \gamma$ (see equation 3). If now a second

line of tangents, A′ B′, be drawn corresponding to the radius $Om = r$ sec. γ, and the distances *am* and *bm*, which are the distances from the extremities of the great circle arc to the foot of the perpendicular from the point of tangency to that arc, be laid off upon it, we shall have the values of angles α and β, whose algebraic sum constitutes the great circle distance. Upon this principle the scales for the measurement of distances on the great circle sailing charts of the U. S. Hydrographic Office are constructed.

MEASUREMENT OF GREAT CIRCLE COURSES ON GNOMONIC CHARTS.

On the gnomonic chart the problem of finding at any point of a ship's track that course which would lead most directly to the point of destination resolves itself into finding the relation between the angle α formed by the straight lines *ab* and *a*P on the plane of projection AB and the spherical angle α_1 formed on the sphere by the great circle arcs a_1b_1 and a_1P_1.

In the figure, let O be the center of the sphere; AB the plane of projection, tangent to the sphere at the point *c*; P_1 the pole of the sphere, projected at P on the plane; a_1b_1 any great circle on the sphere, projected at *ab* on the plane; ψ the angle at the center of the sphere, subtended by the straight line connecting the point of contact with the extremity of the projected great circle.



Suppose the center O and the plane AB to remain stationary and the sphere to be revolved, so that the pole P_1 will be projected at *a* and the

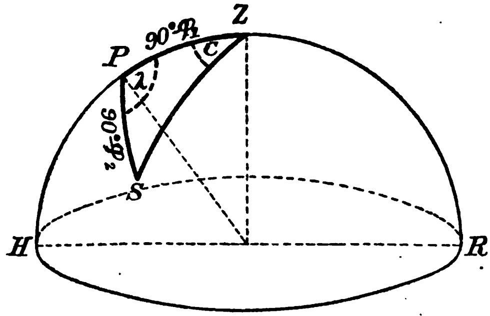
middle meridian P_1c along the line passing through ac. In the new position of the sphere the latitude of the point of contact will be 90° minus the angular distance of c from a. Or, as $ca=r \tan \psi$, c will be in lat. $90°-\psi$.

Regarding now a as the pole, and measuring distances of 90° each, from a to L^o, L', and L'', the points L^o, L', and L'' will be the intersections of the meridians ac, ab, and aP with the projection of the equator in the new position of the sphere; aL^o will be the middle meridian and aL' and aL'' will represent the meridians in longitudes L' and L''. Their difference of longitude $L''-L'$ will be the value of the true angle α_1. Thus it appears that, if a net of the meridian lines of a projection having its point of contact in latitude $(90°-\psi)$ were prepared, the value of α_1 could be readily found by laying the middle meridian of the net over the line ac with the pole over a, and reading off the difference of longitude between the lines aP and ab. But, as the distance ac may vary from 0° to 90°, a great number of such nets would be required to meet every case. A method of combining them was therefore devised, which has resulted in the production of the great circle compasses or course indicators which appear on the great circle sailing charts of the Hydrographic Office. This consists of a series of equidistant concentric circumferences, which, in the gnomonic chart of the Indian Ocean, has for its center the point of contact. Each circumference is divided into 360 degrees, in such a manner that if each division of any circumference, as for example the one marked latitude 36°, were connected by radii with the common center, the complete radiation of the meridians from degree to degree would be shown for a gnomonic projection having its point of contact in latitude (90°—36°) or 54°. The circumferences are marked in the direction of the radius from 3 to 3 degrees for values of ψ, and by reading from the chart the latitude of a the circumference which is to be used in the measurement of the angle α is known at once. Instead of using this course indicator in the manner described for the use of the transparent net of meridians, greater convenience is attained by engraving it in a convenient position on the chart plate and transferring the point from which the course is to be measured to the center of it.

In 1892 a much simplified great-circle course diagram was constructed, and so applied to the series of Great Circle Sailing charts of the Hydrographic Office as to provide for the measurement of great circle courses from a compass-rose in nearly the same manner as on the Mercator chart and with nearly equal convenience.

The problem of finding the initial course on a great circle track passing between two known geographical positions may be stated thus: Given the two sides and their included angle in a spherical triangle to find the angle opposite to a known side. In the following figure let the arc Z S represent a great circle track joining the point of departure Z with the point of destination S, and let P represent the position of the

elevated pole of the earth, then PZ and PS will represent arcs of the meridians of longitude passing through Z and S respectively, the angle ZPS will represent λ, the difference of longitude between Z and S, and, if φ_1 is the latitude of Z and φ_2 the latitude of S, the length of the



sides PZ and PS of the spherical triangle PZS will be $90^\circ-\varphi_1$ and $90^\circ-\varphi_2$ respectively.

From the fundamental formulæ of spherical trigonometry we have

$$\sin(90^\circ-\varphi_1)\cot(90^\circ-\varphi_2)=\cot C\sin\lambda+\cos(90^\circ-\varphi_1)\cos\lambda \quad (1)$$

$$\text{or } \cos\varphi_1\tan\varphi_2=\cot C\sin\lambda+\sin\varphi_1\cos\lambda \quad (2)$$

$$\text{whence } \cot C=\frac{\cos\varphi_1\tan\varphi_2-\sin\varphi_1\cos\lambda}{\sin\lambda} \quad (3)$$

Dividing both terms of the fraction of the second member by $\cos\varphi_1$,

$$\cot C=\frac{\tan\varphi_2-\tan\varphi_1\cos\lambda}{\sec\varphi_1\sin\lambda} \quad (4)$$

$$\text{or } \cot C=\frac{\tan\varphi_2-\tan\varphi_1\cos\lambda}{\sec\varphi_1\sin\lambda-0} \quad (5)$$

$$\text{Putting } \left.\begin{aligned} y&=\tan\varphi_1\cos\lambda\\ y'&=\tan\varphi_2\\ x&=\sec\varphi_1\sin\lambda\\ x'&=o\end{aligned}\right\} \quad (6)$$

equation (5) becomes

$$\cot C=-\frac{y-y'}{x-o} \quad (7)$$

Therefore C is the angle made with the axis of Y by a straight line joining the points (x, y) and (o, y').

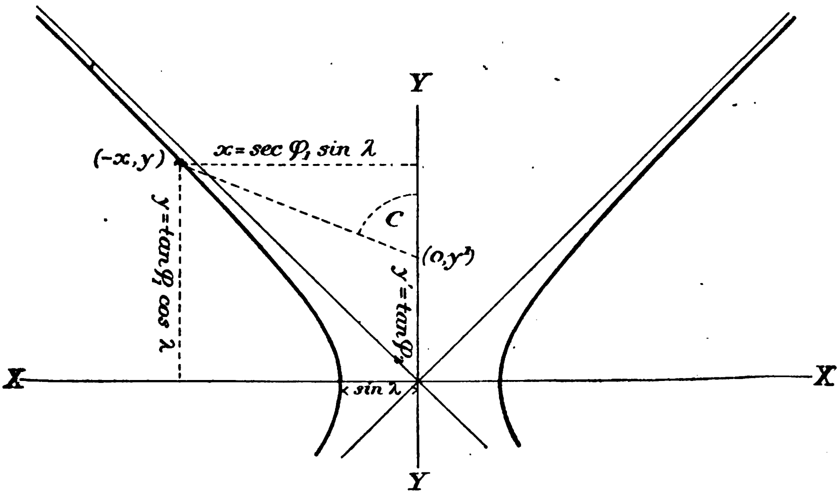
From equation (6)

$$\frac{x}{\sin\lambda}=\sec\varphi_1 \text{ and } \frac{y}{\cos\lambda}=\tan\varphi_1$$

$$\text{or } \frac{x^2}{\sin^2\lambda}-\frac{y^2}{\cos^2\lambda}=\sec^2\varphi_1-\tan^2\varphi_2=1 \quad (8)$$

which is the equation to a hyperbola whose semiminor and semimajor axes are $\sin\lambda$ and $\cos\lambda$, respectively.

Therefore the point (x, y) will always be found upon the arc of a hyperbola upon whose major axis the corresponding point (o, y') is found with reference to a scale of tangents representing $y^1 = \tan \varphi_2$.



In the construction of the course diagram of the accompanying great circle chart of the North Pacific Ocean, λ was assumed to be equal to 20°. This explains why, in the directions for measuring courses, a point is always selected on the great circle track at an interval of 20° of longitude from the point of departure.

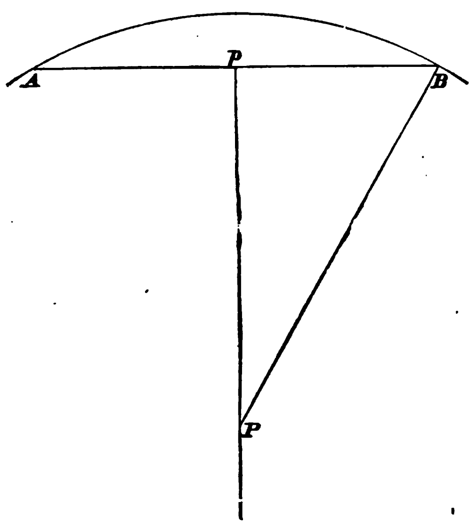
SECTION III.

MISCELLANEOUS METHODS.

AIRY'S METHOD.

Sir George Airy, late astronomer royal of Great Britain, has devised the following graphical method of describing upon a Mercator projection that circular arc which shall approximate most nearly to the projection of a great circle of the globe.

Draw the rhumb-line AB between the place of departure and the place of destination, and the perpendicular *p*P at the middle point of AB. With the middle latitude between the two places enter the following table and take out the corresponding parallel. The center of the required circular arc will be at the intersection of this parallel with the perpendicular.



Middle latitude.		Corresponding parallel.	Middle latitude.		Corresponding parallel.
°		° '	°		° '
20		81 13	52		11 33
22		78 16	54	Opposite name.	6 24
24		74 59	56		1 13
26		71 26	58		4 00
28		67 28	60		9 15
30		63 37	62		14 32
32		59 25	64		19 50
34	Opposite name.	55 05	66		25 09
36		50 36	68		30 30
38		46 00	70	Same name.	35 52
40		41 18	72		41 14
42		36 31	74		46 37
44		31 38	76		52 01
46		26 42	78		57 25
48		21 42	80		62 51
50		16 39			

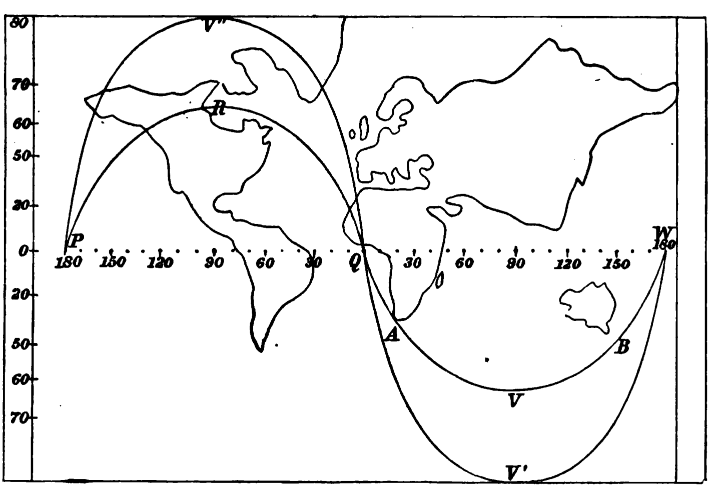
The above table is taken from "Notes on Navigation and the Determination of Meridian Distances," by Commander P. F. Harrington, U. S. Navy.

The center of the arc to be described frequently falls beyond the limits of the chart which may be employed, and the track may occupy more than one chart. In these cases difficulties arise in drawing the required arc.

It is of value to project a great circle track to which it is intended to adhere throughout a passage, but if a vessel be diverted from the projected great circle a new track must be laid down with the attendant inconveniences. Great circle distances can not be measured closely on the approximate arc.

FISHER'S METHOD FOR CIRCULAR ARC SAILING.

Besides the foregoing there is another method for describing circular arcs as substitutes for actual great circle tracks on sailing charts. This method was proposed by the Rev. George Fisher, M. A., F. R. S., and described in the article on circular arc sailing in Riddle's Navigation, published in 1864. Unfortunately it finds its most extensive and easy application in those cases in which the vertex of the great circle is in so high a latitude as to make the navigation of it dangerous. In the tropics the method is impracticable, because great circular arcs are there represented on sailing charts nearly as straight lines; and in the region between 40° and 60° of latitude the curvature of the great circular arcs on sailing charts is so small that it is difficult to describe substitutes for them on account of the length of the radius required.



The curve PRQAVB shows the direction of the great circle which passes through the points A and B, which represent the Cape of Good Hope and the south part of Van Diemen's Land. Although in sailing

from one place to another the navigator is only concerned with that portion of the great circle upon which he means to travel, the curve is nevertheless continued through both hemispheres in the figure, in order that its general form may be the better comprehended. The northern and southern portions of this curve are equal and similar to each other; and the curve cuts the equator at two points, P and Q, at a distance of 180° of longitude from each other, and at an angle, which, at the point of intersection or contrary flexure, is equal to the latitude of the vertex or the inclination of the planes of the equator and the great circle. In like manner the curve PV″ QV′ represents another great circle projected upon the chart, the highest point of latitude V″ or V′ being 80°.

The near approximation to a circular form of the portion of the curve which is nearest to the vertex affords an easy and simple mode of delineating upon a Mercator chart a ship's route between two places thus situated.

It consists in finding the position of the vertex of the great circle which passes through the points of departure and destination, and describing as the track to be pursued a circular arc passing through these three places. Upon this arc the geographical positions of any number of points can be determined to a degree of accuracy which will depend upon the scale of the chart, which known points can be sailed toward in succession by either using Middle Latitude or Mercator's Sailing.

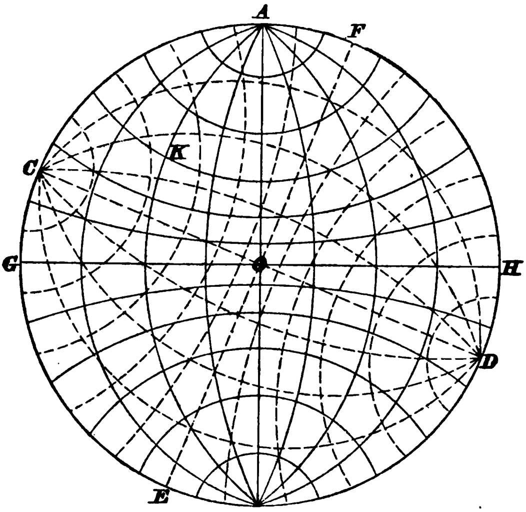
When the great circle track passes to a higher latitude than the navigator wishes to attain, Mr. Fisher proposes a circular track, which shall pass through the three points representing the places of departure, destination, and the highest point which it is desirable to attain.

In the practical application of this method, instead of drawing the whole of a circular arc between the places of departure and destination, it is necessary only to assume one point upon the arc at a convenient distance from the place of departure. Then determine its position upon the chart, and shape a course toward it. Should the ship's position by subsequent observations be found to be off the circular arc previously determined, instead of trying to return to it another circular arc should be described upon the chart extending from the ship's position to the place sailed for, and another point taken upon, as before.

CHAUVENET'S GREAT CIRCLE PROTRACTOR.

By the use of this protractor the latitudes and longitudes of all points on the globe through which the great circle route passes, and also the course to be steered and the distance to be sailed are found by inspection. It consists of two stereographic projections of the spherical rectangular co-ordinates on the same scale, one fixed and the other transparent, and designed to revolve concentrically with the first. The fixed projection represents the meridians and parallels of latitude of the globe to every degree. No land is shown, so that any meridian may be

assumed as the meridian of the place of departure. For simplicity the primitive or bounding meridian is always so taken. In the figure the



full lines and the lines of dashes represent the lines on the fixed and transparent projections, respectively. The lines corresponding to the meridians on the revolving projection evidently represent a complete system of great circles with reference to CD as the equator, while the lines corresponding to the parallels are lines which divide the great circles of the system into sections of equal length, and hence may be called distance lines. The line GH of the fixed projection is graduated from zero at G and zero at H to 90° at O, and forms the scale of longitudes. The corresponding line EF of the revolving projection is similarly graduated, and forms the scale of courses. Since the inclination at C of each meridian to the primitive must be equal to its longitude from the primitive, the line AO of the fixed projection is graduated from zero at O to 90° at A and at B. The corresponding line CO of the revolving projection has a double graduation proceeding from 90° at O to zero and 180° at C and at D.

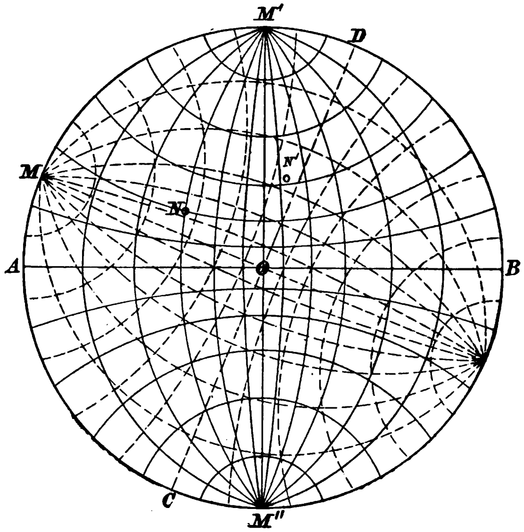
To find the great circle route between any two places, C and K, on the fixed projection, plot the position of C, the point of departure, on the primitive meridian by means of its latitude, and the position of K in its proper latitude on that meridian of the fixed projection, whose angular distance from the primitive is equal to the difference of longitude between C and K. Revolve the transparent projection until the western common intersection of the system of great circles coincides with the point of departure if that point is on the western side of the fixed chart, or until the eastern common intersection of the system of great circles coincides with the point of departure if it is on the eastern side of the fixed chart. The great circle which passes through K is the one re-

quired. The reading of its intersection with EF (see figure) is the first course. The distance in nautical miles between any two points on the great circle is found by observing the distance lines which pass through these points and multiplying the difference of their readings on the scale CD (see figure) by 60′.

In some positions of the protractor such a maze of lines is formed as to give rise to great indistinctness and consequent difficulty in making the required reading. This, added to the liability of the transparent material to become warped and brittle, and the difficulty of keeping the two projections concentric, has caused its disuse.

SIGSBEE'S GREAT CIRCLE PROTRACTOR.

It has been pointed out that the objectionable feature of Chauvenet's Great Circle Protractor is the transparent revolving projection. In the present method, which was produced by Commander C. D. Sigsbee, U. S. Navy, in 1885, this defect has been overcome by the adaptation of a single projection or system of circles to the measurement of more than one system of co-ordinates in the solution of the same problem. This method is capable of the approximate solution of spherical problems in general, but it will be treated of here, so far as it relates to the solution of problems in great circle sailing only. The following description is transcribed from an article entitled "Graphical Method for Navigators," by Commander Sigsbee, U. S. Navy, printed in the Proceedings of the United States Naval Institute, Vol. XI, No. 2, 1885:



In the figure, let us suppose that the full lines, consisting of a system of great circles and parallels on the stereographic projection, with M′ and M″ as the poles of the latter, have served to project the points M and N, the co-ordinates of which are given.

Let us suppose further that M is one of the poles of a similar set of great circles, and that to this system of dotted lines, which serve for the measurement of another system of spherical co-ordinates, the point N is to be referred for the solution of the problem. To make the explanation more practical, let us assume first, that the full lines consist of vertical circles or circles of azimuth and parallels of altitude, the *primitive* or bounding circle being the meridian of the observer, M′ his zenith, and that having the azimuth and altitude of a heavenly body given, we have projected its place, N, upon the figure; second, that M is the elevated celestial pole, the dotted lines hour circles and parallels of declination, and that we require the hour angle and declination of the body at N.

If the graduations of the primitive and of the dotted diameter or equinoctial CD were properly numbered we would simply have to note the parallel of declination and the hour circle passing through the point N, follow one to the primitive and the other to the equinoctial, and take readings, in order to find the declination and the hour angle of the body, that is to say, the co-ordinates of the point N, according to the system of co-ordinates measured by the dotted lines. So long as the relative positions of the two points M and M′ remained the same such a figure would serve for the solution of similar problems involving any other position of an interior point, N; but, since the relative positions of M and M′ are constantly changing in practice, no two sets of lines similar to those of the figure, and printed upon a single sheet, can be of universal application in the manner described. The object of the present method is to overcome this difficulty by adapting the system of full lines to serve the purpose of both systems for all positions of M and N.

The position of N, with respect to the dotted lines, is defined by its position relative to the points M and O. Since the two systems of lines are similar, if we transfer N to N′ so that N′ shall have the same position with respect to M′ and O that N has with respect to M and O, then N′ will have the same relation to the full lines that N has to the dotted lines, and we may therefore let N′ represent N, and the system of full lines, in connection therewith, represent the system of dotted lines. Briefly, then, the method is to assume, first, that the full lines represent a system of spherical co-ordinates to correspond with the given data, and then to project M and N; next, to transfer N to N′, and then to assume that the same lines represent another system of spherical co-ordinates, to which it is necessary to refer to N′ for a solution.

The following elementary, graphical process forms the basis of solutions. Having projected on the diagram two points, as M and N, given in position, one upon the primitive or bounding circle, and the other within, conceive a sector, MOC, whose radii, OM and OC, shall include these points. Conceive the imaginary sector to be revolved about O until M coincides with some other given point upon the primitive, as

M′; then find N′, the revolved position of N. *The radii need never be drawn.*

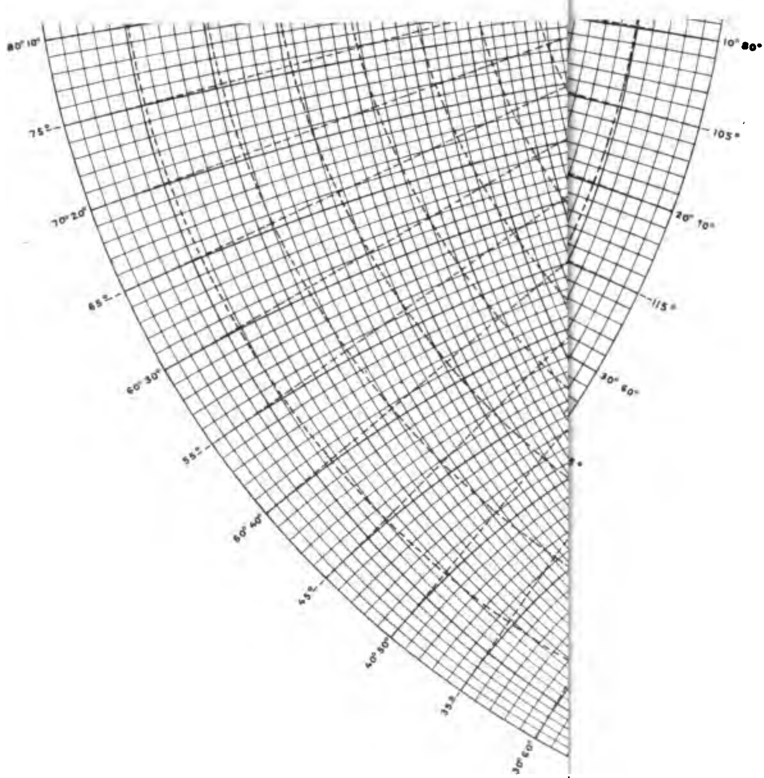
There are various ways of finding N′, but the following are suggested. The first is always available, and involves marking points only upon the diagram; the second requires a piece of tracing-paper, but makes no marks upon the diagram. Since one case embraces all, let it be required to revolve the sector MOC about O until M coincides with M′, and find N′. Since M will traverse the arc MM′, the point C will traverse an equal arc CA.

First method.—Align a straight edge on O and N to find the point C. Make the arc CA equal to the arc MM′, either by the divisions of the scale on the primitive or by transferring the chord MM′ to CA with a slip of paper. Align a slip of paper on O and C and mark upon it the points O, N, and C. Then align the slip on O and A so that its marks O and C shall coincide with O and A of the diagram, respectively. The point N of the slip will be N′, the revolved position of N.

Second method.—Lay a piece of tracing-paper upon the diagram and trace the points M, O, and N. Revolve the tracing about O until M coincides with M′; the traced point N will fall at N′.

The adaptation of the method to great circle sailing.—Let M be the place of departure, and N the place of destination. To find the great circle course, first assume the diagram to be the projection of the terrestrial sphere, composed of parallels of latitude and meridians of longitude. M′ is the north pole, M″ the south pole, and AB the equator. The primitive is always the meridian of departure. Project M upon the primitive in its proper latitude, north or south, as the case may be—on the right side if it is the eastern place, on the left side if it is the western place. Project N in its proper latitude, and upon a meridian whose difference of longitude from M is that of the two places. Conceive a sector, MOC, formed by radii, to include M and N. Note, by a glance simply, if N would fall above or below AB if the sector were revolved so as to make M coincide with that extremity of AB which is adjacent to M. If above, reckon the course from north; if below, from south. Revolve the sector about O until M coincides with M′ or M″, the nearer extremity of M′ M″, and find N′, the revolved position of N. Now, assume M′ or M″, whichever is the revolved position of M, to be the place of departure and N′ the place of destination. The former meridians then become great circles through the place of departure, and the parallels are parallels of great circle distance from the same place The scale AB gives the angle which each great circle makes with the primitive, the meridian of departure, and hence the course.

The great circle passing through M′, N′, and M″ is the required great circle. Read the course at its intersection with AB, reckoning from the nearer extremity of AB. Having the course, reckon it from north or south, as previously found, and towards the east or west, as the place of destination is to the eastward or westward.



80° 10°
75°
70° 20°
65°
60° 30°
55°
50° 40°
45°
40° 50°
35°
30° 60°
10° 80°
105°
20° 70°
115°
30° 60°

To find the great circle distance.—Note the reading upon the primitive at the parallel of distance passing through N′, reckon the distance from M′ or M″, the place of departure, by taking the complement of the reading. Multiply the degrees by 60 and add the minutes; the result will be the distance in nautical miles.

To find the vertex and other points upon the great circle.—The quickest method is by means of tracing paper. Trace the required great circle through N′, and revolve the tracing about O until M′ or M″, whichever is the revolved position of M, coincides with M. The traced great circle will then pass through M and N. If the vertex is of any use it falls upon the diagram, and it is found upon a meridian at 90° difference of longitude from the point where the traced and revolved great circle intersects AB, the equator.

Take points upon the revolved great circle at 5° or 10° intervals of longitude from M towards N, or, if desired, on both sides of the vertex when it falls between M and N, and find the latitude and longitude of each, measuring latitudes upon the primitive and *differences of longitude from* M upon the scale of AB. Transfer the points to the sailing chart and adjust or "fair" a curve to them.

The more exact method is to take the intervals from M toward N, for the points will then fall upon printed meridians of the diagram. The advantage of measuring from the vertex is that points equally distant in longitude on either side have the same latitude.

HARRIS'S* METHOD OF USING A STEREOGRAPHIC PROJECTION IN THE SOLUTION OF SPHERICAL TRIANGLES.

1. One of the difficulties in getting close results with either Chauvenet's or Sigsbee's protractor, and with most planispheric arrangements, arises from the fact that the paper or other material upon which the projections are printed may not contract or expand uniformly in all its parts. For instance, one diameter of the printed hemisphere may be a degree or two longer than the diameter perpendicular to it. Any process which involves *actual* rotation, about the center of the projection, of any point or line is objectionable unless the projection be made upon a plate of metal. If an instrument of this kind were contemplated, a pair of graduated arms radiating from the center and capable of being set at any angle, but which could be removed at pleasure, would be found to be convenient.

The protractor here described aims to do away with the objections just raised by adding a set of dotted lines, representing meridians and parallels of an equatorial projection, to the ordinary meridional projection of the hemisphere. See Fig. 1, in which the broken lines are intended to consist of dots one degree apart. If, therefore, the paper becomes somewhat distorted, the relation between the two sets of lines or coordinates will remain unchanged.

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*Due to Dr. Rollin A. Harris of the U. S. Coast and Geodetic Survey.

2. *Right-angled triangles.*

Always suppose the right angle to fall upon the bounding great circle, say at R, Fig. 2. (Reference to Figs. 2, 3, or 4, implies reference to Fig. 1 also.) Suppose one of the oblique angles to be placed at the pole N; when one of these two angles is known, place there the known angle.

The value of the angle is denoted by the numbering of the great circles. The length of the leg along the limb of the projection is denoted by the numbering of the graduations or divisions; the length of the other leg, by the numbering of the dotted small circles (along AB, Fig. 1).

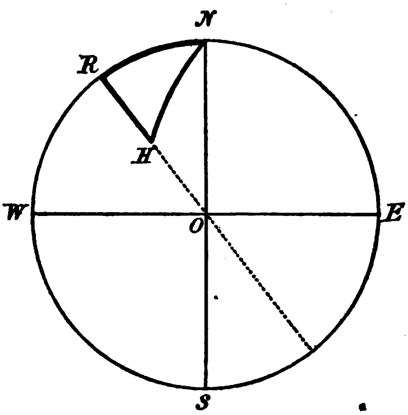

Fig. 2.

This construction suffices for all right triangles, excepting the case where the two given parts are the oblique angles. In this case subtract each of them, as well as the right angle, from 180°. These three supplements are the three sides of the polar triangle. It is a quadrantal triangle and falls under §3; two angles being thus found, their supplements are the two legs of the original triangle.

3. *Quadrantal triangles.*

Let the line ON, Fig. 3, represent the side whose length is a quadrant. Let NH and HO represent the other sides. The numbering of the dotted small circles (Fig. 1) which measure HO should increase from the center outward; at the circumference the numbering is 90°. When the side to be taken along OR exceeds 90°, the numbering increases from 90° at the circumference to 180° at the center, just as if this line were continued on the under surface of the paper. The angle at O is measured by the arc NR; the angle at N is measured along OW, the center O being numbered 0° and 180°, the point W, 90°.

This construction suffices for all quadrantal triangles, except where the angle opposite the quadrant is one of the parts considered. This case can be treated by taking each known part, including the quadrantal side, from 180°, thus obtaining the polar triangle. This being a right triangle falls under §2.

4. *All cases of oblique triangles, excepting where the three sides or the three angles are the given parts.*

Form two right triangles out of, or in connection with, the given

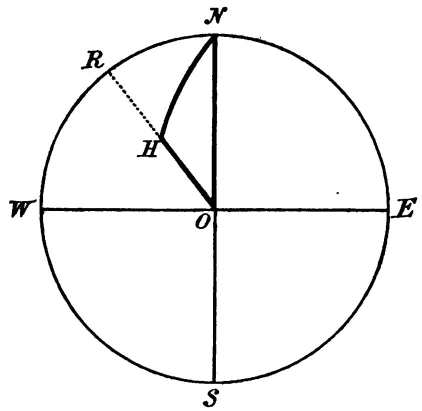

Fig. 3.

oblique triangle by letting fall a perpendicular arc from one of the angles not immediately sought, upon the opposite side, or side produced.

Treat the two right triangles by § 2.

5. *Special treatment for the case where two sides and the included angle are given.*

Let NG, Fig. 4, denote one of the known sides. The known angle

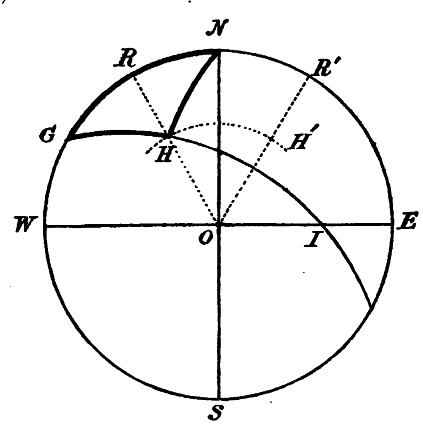

Fig. 4.

at N shows along what arc the second known side NH is to be taken. The dotted radial line OR passing through H has a definite number NR assigned to it. If we imagine GH to be rotated about O until G falls at N, the angle ENH′ will be equal to the original angle NGH. The arc NH′ will be equal to the original unknown side GH. Instead of actually rotating H, the position H′ is found by means of the fixed

numbers belonging to the radial lines, say, by counting from H to H′ a number of degrees equal to GN.

6. *To find the node or vertex of the great circle passing through two given points.*

In the right triangle GWI, the side GW and the angle HGW are known by the preceding solution of the triangle NGH; this being a right triangle, comes under § 2. WI denotes the longitude of the node from W, and the angle HIW, the latitude of the vertex. The longitude of the vertex is, of course, 90° from I.

7. *To construct the track.*

For ascertaining the points through which the great circle track passes, imagine the dotted lines of Fig. 1 to represent meridians and parallels. The system of great circles (full lines) represent paths intersecting at all possible angles with the equator. The orthogonal system of small circles serve as distance lines. The longitude of the node having been found as above, also the latitude of the vertex, each point of the track becomes known in latitude, longitude (from the node), and distance. The bearings of the intermediate points are the angles made by the track and the dotted meridian lines; hence they can be roughly measured or estimated. When both the northern and southern hemispheres are concerned in the same problem, we can imagine the same projection or sets of lines on the under side of the paper as on the visible side.

8. *Given the three sides of a triangle to find two of the angles.*

In order to facilitate the solution of the problem it may be well to make use of a smaller stereographic projection than Fig. 1 for obtaining the first approximate results. In connection with this there should be used a transparent sheet upon which any given small circle can be traced (Sigsbee's method), or upon which is traced, once for all, the entire projection (Chauvenet's method).

Lay off the side opposite the angle not sought along the bounding circle of the protractor, counting distance from one of the poles. Mark the two distance lines (small circles) whose numbers denote the lengths of the two remaining sides. Now if one of these distance lines be traced upon tracing paper, and if the line of centers (central meridian) be rotated until it comes to the other extremity of the side laid down along the bounding circle, the point in which the two distance lines intersect will be determined. The number of the meridian passing through this point shows the value of the angle opposite the side corresponding to the rotated distance line. By marking this point on the traced distance line, and rotating it back to the initial position, the meridian (counted from the other limb of the projection) passing through this point will denote the other angle, which lies upon the bounding circle.

The more accurate values are obtained by means of the large protractor, Fig. 1, and a small piece of tracing paper. From the small protractor, if a small one has been used, the approximate position of

the point of intersection rotated back, becomes known. Place a piece of tracing paper over the large protractor, and trace upon it a few dots of the dotted lines and a small arc of the distance line. Rotate it by addition, or by counting an amount equal to the number of degrees in the side laid off on the bounding circle. Then mark upon it accurately the point of intersection of the rotated and stationary distance circles. The numbering of the great circle passing through this point will denote one of the required angles. Carry the arc thus marked back to its original position. This will give the other angle.

Whenever only rough results are required, a rather small Chauvenet's protractor is a great convenience in almost all problems involving spherical triangles. For instance, a heavenly body whose hour angle and declination are given, together with the latitude of the place, can be immediately referred to alt-azimuth coordinates by inclining the axes of the projections to each other an angle equal to the colatitude of the place. Such problems are solved by this means in a very natural manner. In fact, this small protractor will be found to be highly serviceable in laying out the work to be performed upon the large projection, or in roughly checking the results.

9. *Given the three angles of a triangle to find two of the sides.*

Subtract each angle from 180°, thus obtaining the lengths of the sides of the polar triangle. Two angles of this triangle become known by the preceding method. These angles subtracted from 180° give two of the required sides of the original triangle.

PROCTOR'S METHOD.

By a comprehension of the conservative disposition of seafaring men, and by a study of their wants, the late Richard A. Proctor arrived at a method for finding the shortest sea routes, which, comprising nearly the whole of the navigable world within small compass, is unsurpassed for the generality of the results obtainable.

Proctor was evidently unaware of the successful issue of the investigations into the methods for measuring courses and distances on the gnomonic projection. Indeed, these results have only lately appeared on the great circle sailing charts of the U. S. Hydrographic Office, and the mathematical theory of them is given for the first time in this work.

Though searching in his investigations for some projection on which the great circle will appear in simple geometrical form, he discarded the gnomonic partly on account of the irregular distortion, which precludes the possibility of directly measuring courses by means of the ordinary compass rose or a protractor, and partly because he aimed to show the whole navigable world on one projection, while on a single gnomonic projection it is not possible to show one entire hemisphere.

Like other investigators into this subject, he saw how deep rooted in the maritime world are the ways of handling the Mercator chart, and he aimed to produce a method which would afford as great convenience

for pursuing a great circle track as the Mercator chart affords for pursuing a rhumb. We therefore find him seeking a projection which shall be perfectly correct in detail, one in which distances over small areas are so correctly proportioned that no distortion can be detected, and in which all bearings and directions are correspondingly correct. He was accordingly led to propose the stereographic projection, whereby (1) the construction for marking in the great circle track between any two points is made exceedingly simple; (2) the whole track is obtained at once; (3) the course at each point of the track is shown as plainly as the bearing of the rhumb-line on a Mercator chart; (4) the composite track, where wanted, can be obtained by a simple construction; and (5) the distance from point to point, if wanted, can be easily determined.

In the stereographic projection of the sphere the point of projection O, Fig. 1, is on the surface of the sphere, at the extremity of a diameter PCO, through P, the center of projection. Thus, if *d*P*f* represent the tangent plane through P the points A and B on the sphere would be projected on *d*P*f* at *a* and *b*, where OA and OB produced meet *d*P*f*. If D and E are 90 degrees from P (as in Fig. 1) their projections fall on

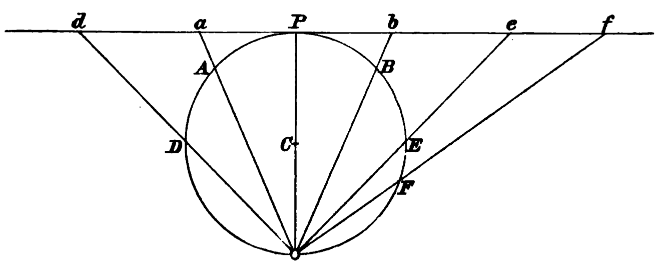

FIG. 1.

*d*P*f* at *d* and *e*. A point, as F, still nearer to O, will be projected on the tangent plane, as at *f*.

If P be either pole the projection of the sphere on the plane *d*P*f* is a very simple matter; for all the meridians are projected into straight lines through P, and all the latitude-parallels into circles around P as center. The radii of these circles can be obtained by construction, as shown in Fig. 1. But in practice it is far better to use their known lengths, as indicated in trigonometrical tables. Thus, if PB is an arc of 60° we know that the angle POB contains 30°, so that P*b* is equal to PO tan 30°. Thus, for the parallels corresponding to latitudes 85°, 80°, 75°, and so on, we take from the trigonometrical tables the natural tangents of $2\frac{1}{2}°$, 5°, $7\frac{1}{2}°$, and so on; and these numbers, with any convenient unit of length, give us the radii of the circles we are to describe round P. For instance, if we wish the equator to have a radius, P*e*, (equal to PO) 5 inches in length, we draw a line 5 inches long, divide it into ten equal parts, and one of these again into ten parts (or preferably we make a plotting scale for the smaller divisions); then, regarding one

of the tenths of the line, *i. e.*, one-half an inch, as our unit, we take for our successive radii lines having the following lengths:

For latitude 85°, 0.437, which is the tangent of $2\frac{1}{2}°$
For latitude 80°, 0.875, which is the tangent of 5°
For latitude 75°, 1.317, which is the tangent of $7\frac{1}{2}°$

and so on. Then a series of radial lines drawn to divisions 5° apart round any one of these circles give the meridians, and complete our projection. The chart should have outlines of continents, islands, etc., marked in for convenience, though in reality this is not essential, because the longitudes and latitudes of ports and places are alone really needed for determining the great circle track; and the track obtained by the simple constructions, which will be indicated, could always be plotted in on the Mercator chart, to the use of which seamen are more accustomed than to that of any other kind of chart.

The properties of the stereographic projection, which enables us at once to project a great circle course and to determine bearings, distances, etc., on a stereographic chart, are the following:

(*a*) Every circle on the sphere, great or small, is projected into a circle.

(*b*) All angles, bearings, etc., on the sphere are correctly presented in the projection (a property found also in Mercator's projection).

With these properties are combined the following properties of great circles on the sphere:

(1) Since every diameter of a great circle is a diameter of the sphere each point on a great circle is antipodal to another point on the same circle, or, in other words, if a great circle passes through any point it passes also through the antipode of that point.

(2) If a great circle touches a small circle on the sphere, it touches also the small circle antipodal to the former; for instance, if a great circle touches latitude parallel 30° north it touches also latitude parallel 30° south. This needs no demonstration, being really a corollary of (1); for the point in which the great circle touches one small circle has for its antipode a corresponding point on the antipodal small circle, and also, by (1), on the great circle; and there can be no other point in which the great circle meets the antipodal small circle, for if there were then, by (1), there wo uld be corresponding points of contact or of intersection with the original small circle, which, by our hypothesis, is not the case.

To use the charts, however, the seaman need not concern himself either with the method of constructing them or with the principles on which their use in great circle sailing depends. All he need care for is rightly to apply the constructions which result from these principles.

It is proposed to indicate only what are the processes necessary for the five following problems:

I. *To find the great circle track between any two points, as* A *and* B, Fig. 2.

II. *To find the vertex* V, *or highest latitude reached on that track.*

III. *To find the bearing at any point, as* Q, *on the track.*

IV. *To find the composite track, from port to port, touching any given limiting latitude.*

V. *To find the distance* AVB *to be traversed.*

The constructions for these five problems are all included in the following simple statements (P, Fig. 2, is the pole, *e*′EE′ the equator), the

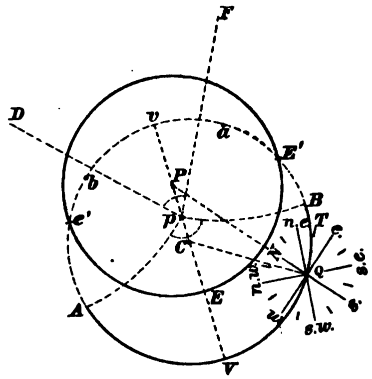

FIG. 2.

italicised parts indicating the actual constructions, the rest giving the reasons and demonstrations:

I. *Find on the chart a and b the antipodes of* A *and* B (which, of course, is easy, as we have only to take off 180° on the meridians AP*a*, BP*b*, not shown in the figure, to avoid crowding); then *a circle through any three of the points*, A, *a*, B, *b*, *is a great circle of the sphere by* (1), *and must pass through the fourth point. Describe such a circle* AB *ab;* AVB *is the great circle course required.* [It will be best to take the three points A, B, and *b*, A being the point of departure. Pencil the bisecting perpendiculars to *b*B, AB (not shown in Fig. 2), intersecting in C, around which point as a center describe in pencil the circle AB*b*. Note whether it passes through *a*, for this serves as a test of the accuracy of the result. You may also note whether the points *e*′ and E′, in which it cuts the equator, are, as they should be, on the extremities of a diameter through P; for two great circles on a sphere necessarily intersect on a diameter of the sphere, and therefore, as P is the projection of the pole of the equator, two such points of intersection must lie on a straight line through P.]

II. *A straight line, v*PCV, *through* P *and* C, *cuts the great circle track in V, the vertex required,* V being the highest latitude on one side of the equator, *v* that on the other. [V does not necessarily fall on the actual great circle course between two points. For instance, QB is the great circle course from Q to B, but V lies outside QB.]

III. *Draw* QT, *tangent to the track, at* Q. *Then the angle* PQT *gives the bearing of the track at* Q *from the due northerly direction* QP. By

observing that QT is at right angles to CQ we can get the bearing without actually drawing QT. Thus, in the case illustrated in the figure, the direction QT is north of due east by an angle equal to CQP, easily measured with a protractor.

IV. Suppose AVB, Fig. 3, the great circle course, to have its vertex V in inconveniently or dangerously high latitudes. *Let* L *on* PV *be the highest latitude which the ship must reach. Take l, antipodal to* L; then, if we *bisect* lL *in* G, and describe round P as center the circle cGc', it is obvious that any circle having its center on cGc', and radius equal to GL or Gl, will touch both the latitude parallels HLK and *klh;* and, by 2,

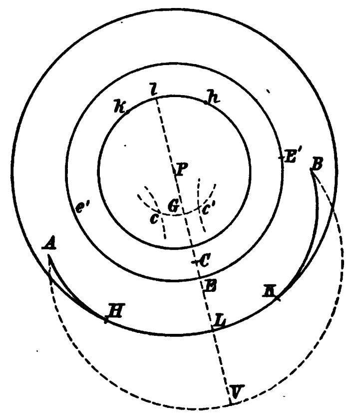

FIG. 3.

will be a great circle. Therefore, *around* A *and* B, *as centers, with radius* GL, *describe circles cutting* cGc' *in c and c'; and with c and c' as centers and the same radius describe circular arcs*, AH, BK, *touching the latitude parallel* HLK *in* H *and* K. *Then* AHKB *is the composite track required.* The distances along AH and BK can be determined by the method shown in the succeeding section, and the distance along the latitude parallel HLK is, of course, easily determined, being an arc of a known number of degrees in a known latitude.

V. [We have to determine how many degrees there are in the arc AVB, not as it appears in the chart, but as it really is on the sphere.] *Take p,* 90° *of latitude, from* V *and v, Fig.* 2, *along* vPV. (This, of course, is done at once on the chart, which shows the degrees of latitude from the pole P.) Then p is the pole of AVB. *Find* D, *the center of the great circle* Ap; *and* F, *the center of the great circle* Bp. (This we do by I.; but most of the work is already done. We have the bisecting perpendicular to Bb; the bisecting perpendicular to Aa passes through C; then the bisecting perpendiculars to Ap, pB give us, by their intersections with those to Aa and Bb, D and F at once.) What we want is to determine *the angle* ApB, between the arcs Ap, pB; and it is obvious that this *is the supplement of the angle* DpF, which is easily

measured with a protractor. *The number of degrees, multiplied by 60, gives the number of geographical miles or knots in the distance* AVB.

An example of these methods is given in the stereographic chart, which has been reduced from Proctor's great circle sailing chart for the Southern Hemisphere. In this chart the same letters are used as in Figs. 2 and 3.

SPHERICAL TRAVERSE TABLES.

Under the methods of Bergen and Towson the general computation of the parts of spherical triangles and the arrangement of the results in the form of spherical tables were treated of. In Raper's excellent work on navigation spherical traverse tables are given. They are adapted to great circle sailing in the same manner as the ordinary traverse tables are adapted to sailing on a rhumb line. Practical rules covering all cases are there given for the guidance of navigators.

GREAT CIRCLE COURSES FROM THE SOLAR AZIMUTH TABLES.

At page 8 of a treatise on Azimuth[1] by Lieut. Commander Joseph Edgar Craig, U. S. Navy, the following equation is stated for the solution of the time-azimuth problem:

$$\cot Z = \frac{\cos L_1 \tan d - \sin L_1 \cos t}{\sin t} \tag{1}$$

from which are derived

$$\tan \phi = \cos t. \cot d \tag{2}$$

$$\text{and } \cot Z = \cot t. \cos (\phi + L_1) . \operatorname{cosec} \phi \tag{3}$$

in which t, d, and Z represent respectively the hour-angle, declination, and azimuth of the observed celestial body, and L_1 the geographical latitude of the observer.

At page 54 of the present work on the Development of Great Circle Sailing, the equations stated for finding the great circle course are:

$$\tan \phi = \cos \lambda \cot L_2 \tag{4}$$

$$\text{and } \cot C = \cot \lambda \cos (L_1 + \phi) \operatorname{cosec} \phi \tag{5}$$

in which L_1 represents the latitude of the ship, L_2 the latitude of the place of destination, λ the difference of longitude between the meridian of the ship and the meridian of the place of destination, and C the course. If, in equations (2) and (3), λ be substituted for t and L_2 for d, their second members will be identical with those of equations (4) and (5), and therefore, when the difference of longitude and the latitude of the place of destination in the great-circle problem are equal respectively to the hour-angle and declination in the time-azimuth problem, the course C resulting from equation (5) will be identical with the azimuth Z resulting from equation (3). Now the values of Z resulting

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[1] Azimuth. A Treatise on this Subject, with a study of the Astronomical Triangle, and of the Effect of Errors in the Data. Illustrated by Loci of Maximum and Minimum Errors. By Joseph Edgar Craig, Lieutenant-Commander, U. S. Navy. New York: John Wiley & Sons, 1887.

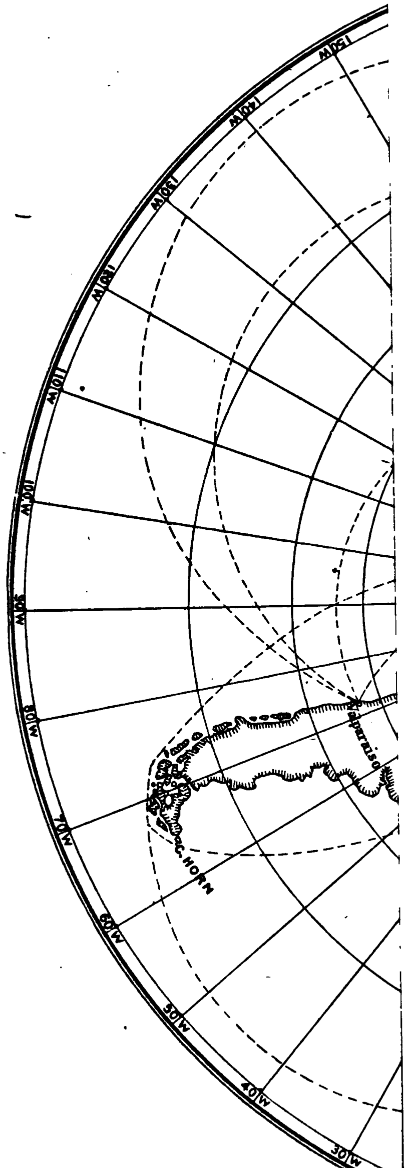

Chart for great circle sailing, sh
the gre

Face page 50.

from equation (3) have been computed for declinations between 23° N. and 23° S., which represent the range of declinations of the sun, and arranged, for stated values of the hour-angle, declination, and latitude, in the Solar Azimuth Tables, which have been well known among mariners for many years. It is therefore evident that, when the port of destination is situated within the Tropics, the Solar Azimuth Tables may be used for ascertaining great circle courses by simply regarding the latitude of the port *bound to* as declination, and the difference of longitude, converted into time, as the hour-angle. The latitude of the ship remains the latitude of the observer as in taking out values of the azimuth from the tables.

The identity of the time-azimuth problem and the great-circle course problem may also be graphically illustrated.

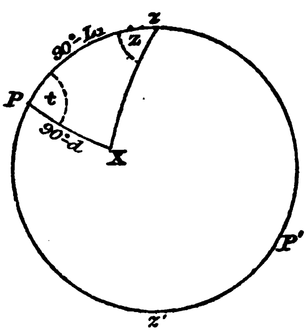

Fig.1.

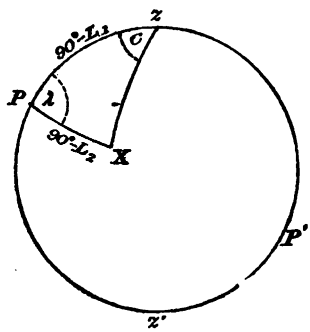

Fig.2.

Let Fig. 1 represent the astronomical triangle, P z X, projected on the plane of the celestial meridian of the observer, Pz P$'$ z', and let P represent the elevated pole, z the zenith of the observer, whose latitude is L_1, and X the position of the observed celestial body, whose declination is d, hour-angle t, and azimuth Z; and let Fig. 2 represent a projection, on the plane of the terrestrial meridian, Pz P$'$ z', passing through the point of departure, of the spherical triangle P z X, whose vertices are the place of departure z, the place of destination X, and the elevated pole P, and whose sides are arcs of the meridian of the place of departure, of the meridian of the place of destination, and of the great circle passing through the places of departure and destination. Then it will be apparent that if the values of L_1 are identical in the two figures, and if λ, the difference of longitude is equal to t, the hour angle, and L_2, the latitude of the place of destination, is equal to d, the declination, the great-circle course C must be identical with the azimuth, Z.

Captain Craig, the author above mentioned, and others of the foremost navigators of the United States Navy, had noticed the ready adaptability of the Solar Azimuth Tables to the finding of great-circle courses, and had for years made use of the knowledge in practical

navigation; but no formal disclosure of the method appears to have been made until reference was made to it in the ninth edition of Lecky's work on navigation, entitled Wrinkles in Practical Navigation. It was the impression of this author, as the Azimuth Tables extend only to 23 degrees of declination, that this method would only be applicable where the latitude of the place of destination is within 23 degrees of the equator, or within the Tropics. It will be valuable, therefore, to point out to navigators that the Solar Azimuth Tables are universally applicable for finding great-circle courses with very great facility, because all great circles pass into the Tropics; and, if the problem of finding the courses is with reference to a great-circle track between a point of departure and a point of destination, both lying outside of the Tropics, it is only necessary to find a point lying on the prolongation of the great-circle arc beyond the point of actual destination and within the Tropics, and treat this point as the place of destination in finding the courses from the Azimuth Tables.*

To illustrate, take the problem of finding the initial course on a voyage from Bergen, in latitude 60° N. and longitude 5° E., and the Strait of Belle Isle, in latitude 52° 12′ N. and longitude 55° W. On a copy of a gnomonic chart, such as Godfray's, which accompanies this work, draw a straight line between the geographical positions above stated and extend it beyond the latter into the Tropics. It will be found to intersect the twentieth degree parallel of latitude in longitude 90° W., or 95° from the meridian of the point of departure. Entering the Azimuth Tables at latitude 60°, under declination 20° and opposite hour-angle 95° or $6^{h}\cdot 20^{m}$, we find the required course to be N. 75° 31′ W.

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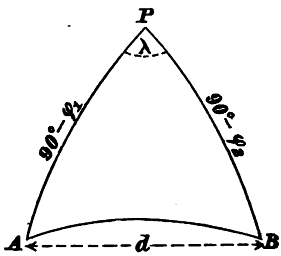
*See "The 'Ex-Meridian' treated as a problem in Dynamics," by H. B. Goodwin, M. A., formerly examiner in nautical astronomy at the Royal Naval College, Greenwich, England. London: George Philip & Son, 32 Fleet street, E. C., 1894.

SECTION IV.

METHODS REQUIRING COMPUTATION.

This section is devoted to the exposition of those methods for finding great circle courses and distances, and the latitudes and longitudes of points on great circular arcs which require computation.

THE COMPUTATION OF GREAT CIRCLE DISTANCES.



Let AB represent the arc of a great circle passing through the two points A and B, whose difference of longitude is λ and whose latitudes are φ_1 and φ_2, respectively. And let d be the distance between A and B, measured along the arc AB. From the fundamental equations of spherical trigonometry we have, in the triangle ABP,

$$\cos d = \cos(90^\circ - \varphi_1)\cos(90^\circ - \varphi_2) + \sin(90^\circ - \varphi_1)\sin(90^\circ - \varphi_2)\cos\lambda$$

$$\cos d = \sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 \cos\lambda$$

$$= \sin\varphi_1 \sin\varphi_2 (1 + \cot\varphi_1 \cot\varphi_2 \cos\lambda)$$

THE COMPUTATION OF GREAT CIRCLE COURSE AND DISTANCE, AND THE LATITUDES AND LONGITUDES OF POINTS ON GREAT CIRCULAR ARCS.

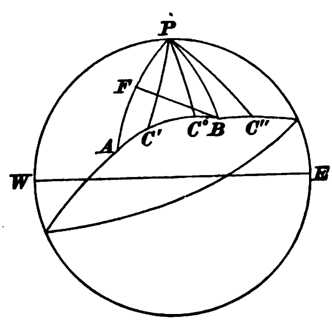


FIG. 1.

Let WE represent the equator, A and B the given points, PA and PB the meridians passing through A and B, respectively, and

AB the required arc. Let BF represent the great circle drawn from one of the given points perpendicular to the meridian passing through the other. Let λ =APB denote the difference of longitude between A and B.

Let L_1 represent the latitude of A,
L_2 the latitude of B,
φ the arc PF.

Then AF will equal $90^\circ-(L_1 + \varphi)$, and we have—

$$\tan \varphi = \cos \lambda \cot L_2 \tag{1}$$

$$\cot C = \cot \lambda \cos (L_1+\varphi) \operatorname{cosec} \varphi \tag{2}$$

$$\cot d = \cos C \tan (L_1+\varphi) \tag{3}$$

in which C represents the course from A; but, should the course from B, be desired interchange L_1 and L_2 in each of the above formulæ. In drawing the diagram, let fall the perpendicular upon the meridian passing through that point from which the course is desired.

The signs of the functions must be carefully noted. That branch of the great circle which corresponds to $\lambda < 180^\circ$ is sought.

φ may be taken in the first or second quadrant, according to the sign of its tangent; but it will be found convenient to restrict it to a value less than 90°, marking it positive or negative, according to the sign of its tangent.

The latitude of one of the points being regarded as positive that of the other point, when of the opposite name, must be marked negative. Thus, in Fig. 2, PB is numerically $90^\circ+L_2$.

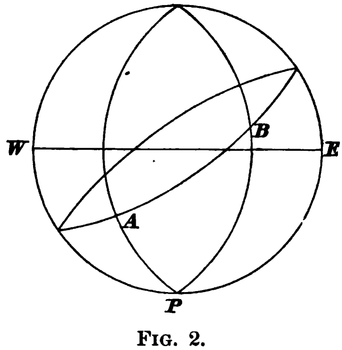


FIG. 2.

To find the position of the vertex, in the triangle PAC° (Fig. 1), C° being the vertex,

$$\cos L_v = \sin C \cos L_1, \tag{4}$$

$$\cot \lambda_v = \tan C \sin L_1, \tag{5}$$

If the angles PAB and PBA are both less than 90°, the vertex will be between the points A and B; but if the course from one given point

is less than 90° and that from the other greater than 90°, the vertex will be upon the arc AB extended. The latitude of the vertex sought will be of the same name as that of the given point, which is nearer to a pole than the other point.

To find other points of the curve, as C′ and C″, assume differences of longitude from the vertex, usually at intervals of 5° or 10°; then

$$\left.\begin{array}{l}\tan L' = \tan L_v \cos \lambda' \\ \tan L'' = \tan L_v \cos \lambda''\end{array}\right\} \qquad (6)$$

Each of these formulæ will determine two points of the curve, symmetrically situated on opposite sides of the vertex; but only one of these will be used when the vertex falls outside of the required arc.

Likewise successive values of the latitudes may be assumed, and the corresponding differences of longitude found by the formulæ:

$$\left.\begin{array}{l}\cos \lambda' = \cot L_v \tan L' \\ \cos \lambda'' = \cot L_v \tan L''\end{array}\right\} \qquad (7)$$

ASMUS'S METHOD FOR THE CONSTRUCTION OF A GREAT CIRCLE ON THE MERCATOR PROJECTION.

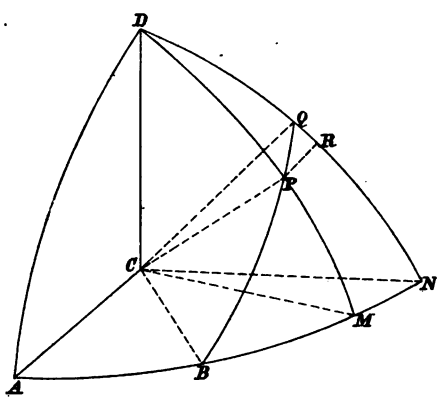


FIG. 1.

Let ABMN (Fig. 1) represent the equator, C the center, $AC = a$ the radius, and D the pole of the earth. Let AD represent the prime meridian, as for instance the meridian of Greenwich; MD, any other meridian; the angle $ACM = \lambda$, the longitude; the angle $MCP = \varphi$, the latitude of the point P.

Let BP represent a great circle cutting the equator at an angle $MBP = \gamma$, and let the point of intersection B be determined by its longitude, β, the angle ACB.

Suppose, now, that BP takes the infinitesimal increment PQ, then λ will be increased by the angle $MCN = d\lambda$, and φ by the angle $RCQ = d\varphi$. PR denotes a part of the parallel passing through P. The radius of the parallel PR is $a \cdot \cos \varphi$.

Since PR is described with the radius $a \,.\, \cos\varphi$, and belongs to the central angle $d\lambda$, the arc $PR = a \,.\, \cos\varphi \,.\, d\lambda$, and since QR is described with the radius a, and corresponds to the central angle $d\varphi$,

$$QR = a \,.\, d\varphi.$$

Since infinitesimal arcs may be considered straight lines it follows that the triangle PQR may be considered a plane one.

In the Mercator projection the lines PR and MN (Fig. 1) are equal to each other. Since this is really not the case, and since it is desirable that the spherical figure and its representation be similar, if not as a whole yet in their smallest parts, and since it is necessary that the angle QPR be the same on the sphere and the chart, we imagine, instead of the triangle PQR a similar one, P′Q′R′ (Fig. 2), whose sides are sec φ times as large as those of PQR. Therefore, in the triangle P′Q′R′,

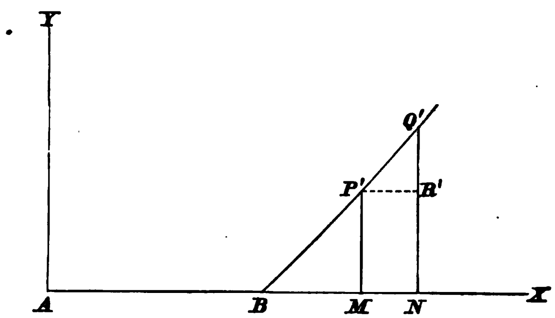


FIG. 2.

$$P'R' = PR \sec\varphi = a \,.\, \cos\varphi \,.\, d\,\lambda \,.\, \sec\varphi = a \,.\, d\lambda$$

which makes $P'R' = MN$,

$$Q'R' = QR \sec\varphi = a \,.\, d\varphi \,.\, \sec\varphi = a \,.\, \frac{d\varphi}{\cos\varphi}$$

In Fig. 2, AB, MN, and AM are the same as in Fig. 1.

$$AB = a \,.\, \beta$$

$$AM = a \,.\, \lambda$$

$$MN = a \,.\, d\lambda$$

Moreover, the triangle P′Q′R′ is similar to the triangle PQR. If the equator be taken as the axis of X, and the prime meridian as the axis of Y, we have:

$$x = AM = a \,.\, \lambda \tag{1}$$

and if the unknown ordinate MP′ be denoted by y,

$$dy = Q'R' = a \,.\, d\varphi \,.\, \sec\varphi = \frac{a \,.\, d\varphi}{\cos\varphi}$$

and by integration,

$$y = a \log \text{nat.} \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right) \tag{2}$$

A third equation is needed which shall express that the great circle BP (Fig. 1) cuts the equator at an angle γ.

From the spherical triangle BMP, two of whose sides are $(\lambda-\beta)$ and φ, we have

$$\tan \gamma \sin (\lambda-\beta) = \tan \varphi \tag{3}$$

Knowing a, β, and γ, and giving x any value, we obtain λ from equation (1), then φ from (3), and finally y from (2).

In order to express the curve BP′ (Fig. 2) only in rectangular co-ordinates, an equation which does not contain the variables λ and φ must be deduced from (1), (2), and (3).

From (1),

$$\lambda = \frac{x}{a} \tag{4}$$

from (2),

$$e^{\frac{y}{a}} = \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$$

or applying the formula

$$\tan 2\theta = \frac{2 \tan \theta}{1-\tan^2 \theta} \quad \text{to the case of } \theta = \left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$$

$$\tan\left(\frac{\pi}{2}+\varphi\right) = \frac{2e^{\frac{y}{a}}}{1-e^{\frac{2y}{a}}} = -\cot \varphi$$

$$\tan \varphi = \frac{e^{\frac{2y}{a}}-1}{2e^{\frac{y}{a}}} = \frac{e^{\frac{y}{a}}-e^{\frac{-y}{a}}}{2} \tag{5}$$

Finally, substituting the values of λ and $\tan \varphi$ from (4) and (5) in equation (3), we obtain as the equation to the curve BP′

$$\tan \gamma \sin\left(\frac{x}{a}-\beta\right) = \frac{e^{\frac{y}{a}}-e^{\frac{-y}{a}}}{2} \tag{6}$$

Up to this point β and γ have been assumed as known. If this is not the case, and we have instead the conditions that the curve must pass through two points whose latitudes, φ_1 and φ_2, and longitudes, λ_1 and λ_2, are given, then β and γ must be computed by the two following equations, which are obtained from (3):

$$\begin{aligned} \tan \gamma \sin (\lambda_1-\beta) &= \tan \varphi_1 \\ \tan \gamma \sin (\lambda_2-\beta) &= \tan \varphi_2 \end{aligned} \tag{7}$$

The quotient of the equations (7) contains but one unknown, which is readily found by following out the operations which are indicated below :

$$\tan\left(\frac{\lambda_2+\lambda_1}{2}-\beta\right)=\frac{\sin(\varphi_2+\varphi_1)}{\sin(\varphi_2-\varphi_1)}\cdot\tan\frac{\lambda_2-\lambda_1}{2}=\tan\delta \tag{8}$$

in which δ is an auxiliary angle.

Having found δ,

$$\beta=\frac{\lambda_2+\lambda_1}{2}-\delta \tag{9}$$

Knowing β, we have from (7),

$$\tan\gamma=\frac{\tan\varphi_1}{\sin(\lambda_1-\beta)}=\frac{\tan\varphi_2}{\sin(\lambda_2-\beta)} \tag{10}$$

In order to construct the curve accurately, β and γ are first determined from (9) and (10), then (1), (2), and (3) are applied to a series of longitudes between λ_1 and λ_2, the extreme longitudes of the track.

In the case in which the terminal points λ_1 φ_1 and λ_2 φ_2 are not far apart the following approximate construction may be applied. For the point of departure we have at once,

$$x_1=a\lambda_1 \qquad y_1=a\log\text{nat}\tan\left(\frac{\pi}{4}+\frac{\varphi_1}{2}\right) \tag{11}$$

and for the point of destination,

$$x_2=a\lambda_2 \qquad y_2=a\log\text{nat}\tan\left(\frac{\pi}{4}+\frac{\varphi_2}{2}\right) \tag{12}$$

Since the straight line between these two points would depart too much from the true curve it is better to connect them by an arc of a circle, to determine which a third point is necessary. This may be most easily obtained by determining the direction of the tangents at the terminal points of the curve P_1 and P_2 (Fig. 3).

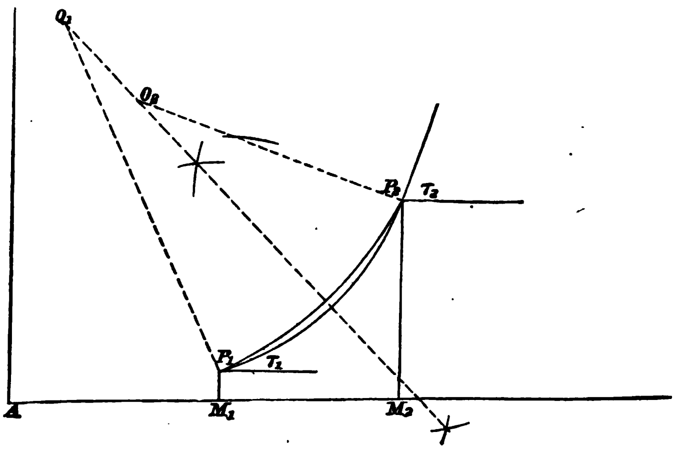


FIG. 3.

Let τ denote the tangent angle=angle RPQ, then

$$\tan\ \tau = \frac{Q^1R^1}{P^1R^1} = \frac{QR}{PR} = \frac{1}{\cos\ \varphi} \cdot \frac{d\varphi}{d\lambda}$$

By differentiating equation (3), we have

$$\tan\ \gamma \cos(\lambda - \beta)\, d\lambda = \frac{d\varphi}{\cos^2\ \varphi}$$

$$\frac{d\varphi}{d\lambda} = \tan\ \gamma\ \cos(\lambda - \beta) \cos^2\ \varphi$$

Whence

$\tan\ \tau = \tan\ \gamma \cos\ (\lambda - \beta)\ \cos\ \varphi$, or since by (3) $\tan\ \gamma = \dfrac{\tan\ \varphi}{\sin\ (\lambda - \beta)}$

$$\tan\ \tau = \cot\ (\lambda - \beta)\ \sin\ \varphi \tag{13}$$

which does not contain γ.

The positions of the tangents at P_1 and P_2 are now determined by the formulæ

$$\tan\ \tau_1 = \cot\ (\lambda_1 - \beta)\ \sin\ \varphi_1 \text{ and}$$

$$\tan\ \tau_2 = \cot\ (\lambda_2 - \beta)\ \sin\ \varphi_2$$

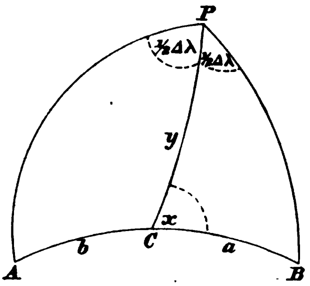
With these two tangents two circular arcs may be constructed passing through the points P_1 and P_2. The centers O_1 and O_2 of these circles are the intersections of the normals at P_1 and P_2 with the perpendicular to the middle point of the chord joining P_1 and P_2. The real curve is contained in the meniscus between the two arcs, and may be drawn sufficiently close with an irregular curve.

This method is too tedious for purposes of navigation, and if used at all by the navigator it will be sufficient to determine the successive positions through which the great circle passes, to plot these positions upon the chart, and connect them by circular arcs. The courses and distances may then be taken directly from the chart. As far as the determination of the positions of the successive points on a great circle is concerned this method is shorter than those ordinarily employed. It originated with J. Asmus, of the Hydrographic office of the German Empire, and was published in the Annalen der Hydrographie und maritimen Meteorologie, 1879, Vol. IV, p. 151 *et seq.*

ZESCEVICH'S METHOD FOR FINDING THE POSITIONS OF THE POINTS OF THE ARC OF A GREAT CIRCLE.

This method, which was published by Professor Gelcich in the "Mittheilungen aus dem Gebiete des Seewesens," Nos. IX and X, 1886, is due to the late German hydrographer Zescevich. It consists in computing the latitude at the middle longitude of a great circular arc, and

has the advantages of simplicity, saving of time in calculation, and ex cluding the possibility of error in plotting.



Let A be the point of departure and B that of destination. Let ACB be the arc of a great circle passing through A and B. Let P be the pole, φ_0 the latitude of A, and φ_1 the latitude of B. AP is therefore equal to $90^\circ-\varphi_0$, and BP equal to $90^\circ-\varphi_1$. Bisect the difference of longitude $APB=\Delta\lambda$ by the meridian PC and let the angle PCB be represented by x.

Let $PC=y$, $AC=b$, and $BC=a$, then, from the triangle PBC,

$$\sin\varphi_1 = \cos y\cos a + \sin y\sin a\cos x \tag{1}$$

$$\cos a = \cos y\sin\varphi_1 + \sin y\cos\varphi_1\cos\tfrac{1}{2}\Delta\lambda \tag{2}$$

Substituting the value of cos a from equation (2) in equation (1), we obtain:

$$\sin\varphi_1\sin y = \cos y\cos\varphi_1\cos\tfrac{1}{2}\Delta\lambda + \sin a\cos x$$

and dividing by $\cos\varphi_1$

$$\tan\varphi_1\sin y = \cos y\cos\tfrac{1}{2}\Delta\lambda + \frac{\sin a\,.\,\cos x}{\cos\varphi_1}$$

Since

$$\frac{\sin a}{\cos\varphi_1} = \frac{\sin\tfrac{1}{2}\Delta\lambda}{\sin x}$$

$$\tan\varphi_1\sin y = \cos y\cos\tfrac{1}{2}\Delta\lambda + \sin\tfrac{1}{2}\Delta\lambda\cot x \tag{3}$$

Similarly we obtain from the triangle APC

$$\tan\varphi_0\sin y = \cos y\cos\tfrac{1}{2}\Delta\lambda - \sin\tfrac{1}{2}\Delta\lambda\cot x \tag{4}$$

Adding (3) and (4),

$$\sin y\,(\tan\varphi_0 + \tan\varphi_1) = 2\cos y\cos\tfrac{1}{2}\Delta\lambda$$

or

$$\frac{\sin y\sin(\varphi_0+\varphi_1)}{\cos\varphi_0\cos\varphi_1} = 2\cos y\cos\tfrac{1}{2}\Delta\lambda$$

or

$$\tan y = \frac{2 \cos \tfrac{1}{2}\varDelta\lambda \cos \varphi_0 \cos \varphi_1}{\sin (\varphi_0 + \varphi_1)} \tag{5}$$

y is the complement of the latitude of the point C, which lies on the great circle equidistant in longitude between the two points marking the extremities of the arc in question. If the latitude of this point be denoted by $\varphi_{\frac{1}{2}}$, then $(90^\circ - \varphi_{\frac{1}{2}}) = y$, and if this value be substituted in (5) and, at the same time, the reciprocal of (5) be taken.

$$\tan \varphi_{\frac{1}{2}} = \frac{\sin (\varphi_0 + \varphi_1)}{2 \cos \tfrac{1}{2}\varDelta\lambda \cos \varphi_0 \cos \varphi_1} \tag{6}$$

In this equation the sign of the $\varphi_{\frac{1}{2}}$ will depend upon the sign of the numerator. If north latitude be denoted by + and south by — the resulting latitude $\varphi_{\frac{1}{2}}$ will be north when the algebraic sum of φ_0 and φ_1 is positive and south when it is negative.

The latitude of that point on the great circle whose longitude is equidistant from A and C is now found by the same formula, and similarly the latitude of the point midway in longitude between C and B.

According to the adopted notation these formulæ will be

$$\tan \varphi_{\frac{1}{4}} = \frac{\sin (\varphi_0 + \varphi_{\frac{1}{2}})}{2 \cos \tfrac{1}{4}\varDelta\lambda \cos \varphi_0 \cos \varphi_{\frac{1}{2}}}$$

and

$$\tan \varphi_{\frac{3}{4}} = \frac{\sin (\varphi_{\frac{1}{2}} + \varphi_1)}{2 \cos \tfrac{1}{4}\varDelta\lambda \cos \varphi_1 \cos \varphi_{\frac{1}{2}}}$$

The latitudes of these points of the great circle, which are midway in longitude between the points now known, can be computed in like manner until the positions of the requisite number of points are established.

In computing these sets of formulæ many of the logarithms are repeated, and the work is thus materially abridged, as may be seen from the following set:

Proceeding with the computation, it would next be necessary to determine $\varphi_{\frac{1}{8}}$, $\varphi_{\frac{3}{8}}$, $\varphi_{\frac{5}{8}}$, and $\varphi_{\frac{7}{8}}$, for which the equations are

$$\tan \varphi_{\frac{1}{8}} = \frac{\sin (\varphi_0 + \varphi_{\frac{1}{4}})}{2 \cos \tfrac{1}{8}\varDelta\lambda \cos \varphi_0 \cos \varphi_{\frac{1}{4}}}$$

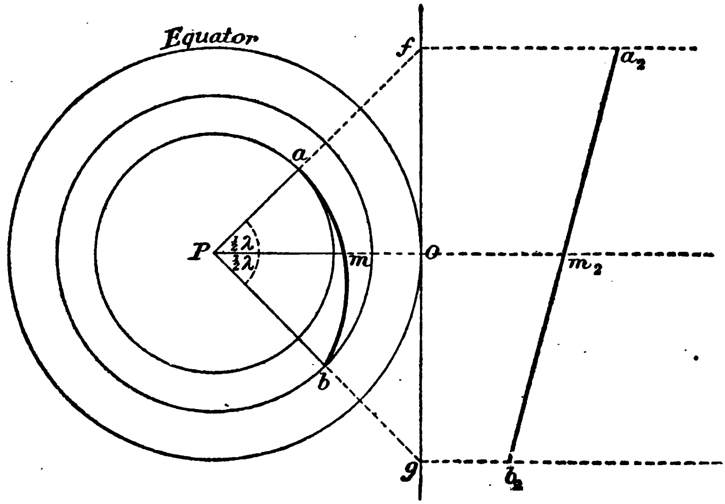
$$\tan \varphi_{\frac{3}{8}} = \frac{\sin (\varphi_{\frac{1}{4}} + \varphi_{\frac{1}{2}})}{2 \cos \tfrac{1}{8}\varDelta\lambda \cos \varphi_{\frac{1}{4}} \cos \varphi_{\frac{1}{2}}}$$

$$\tan \varphi_{\frac{5}{8}} = \frac{\sin (\varphi_{\frac{1}{2}} + \varphi_{\frac{3}{4}})}{2 \cos \tfrac{1}{8}\varDelta\lambda \cos \varphi_{\frac{1}{2}} \cos \varphi_{\frac{3}{4}}}$$

$$\tan \varphi_{\frac{7}{8}} = \frac{\sin (\varphi_{\frac{3}{4}} + \varphi_1)}{2 \cos \tfrac{1}{8}\varDelta\lambda \cos \varphi_{\frac{3}{4}} \cos \varphi_1}$$

THE COMPUTATION OF THE LATITUDE AT THE MIDDLE LONGITUDE.

A shorter and more direct method of arriving at Zescevich's results is as follows: Let the great circle ab be gnomonically projected upon a plane tangent to the sphere at a point O on the equator, as indicated in the accompanying figure, in which the plane of projection is represented as having been rebatted into the plane of the equator.



Let OP $= r = 1$ represent the radius of the sphere,
φ the latitude of the point a, projected at a_2,
φ' the latitude of the point b, projected at b_2,
$\varphi_{\frac{1}{2}}$ the latitude of the point m, projected at m_2,
λ the difference of longitude between a and b.

Then $of = og = \tan \frac{1}{2}\lambda$

$$fa_{,} = \sec \tfrac{1}{2}\lambda . \tan \varphi$$
$$gb_2 = \sec \tfrac{1}{2}\lambda . \tan \varphi_{,}$$
$$om_2 = \tan \varphi_{\frac{1}{2}} = \frac{fa_2 + gb_2}{2}$$
$$= \tfrac{1}{2} \sec \tfrac{1}{2}\lambda (\tan \varphi + \tan \varphi_{,}).$$

If the extremities of the great circular arc in question have latitudes of different name, the formula for the latitude at the middle longitude, φ being the greater latitude, is $\tan \varphi_{\frac{1}{2}} = \frac{1}{2} \sec \frac{1}{2}\lambda (\tan \varphi - \tan \varphi_{,})$.

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